

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman Email/Phone: Kruckman@gmail.com

Speaker's Name: Ya'acov Peterzil

Talk Title: O-Minimal Ingredients in Proofs of Arithmetical Conjectures

Date: 02/06/14 Time: 1:30 am / (circle one) pm Such as Marin-Mumford and Andre-Oort (I)

List 6-12 key words for the talk: o-minimality, Pila-Wilkie, special points, Ax-Lindeman, fundamental domain

Please summarize the lecture in 5 or fewer sentences: Part 1 of 2. Slides (1-21, pp. 1-120 in attached pdf) with some supporting boardwork. A general scheme/strategy for applications of the Pila-Wilkie theorem on counting points on sets definable in o-minimal theories to "special points" problems. A test case was worked out in detail: the "exponential example", in which special points are torsion points in $(\mathbb{C}^m)^+$

CHECK LIST

(This is **NOT** optional, we will **not** pay for **incomplete** forms)

- Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
 - **Computer Presentations:** Obtain a copy of their presentation
 - **Overhead:** Obtain a copy or use the originals and scan them
 - **Blackboard:** Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
 - **Handouts:** Obtain copies of and scan all handouts
- For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list.
(YYYY.MM.DD.TIME.SpeakerLastName)
- Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

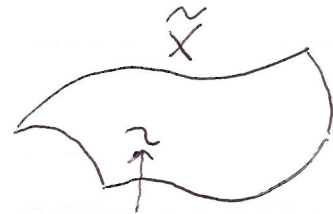
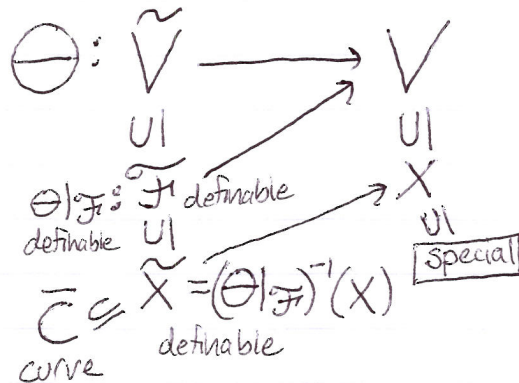
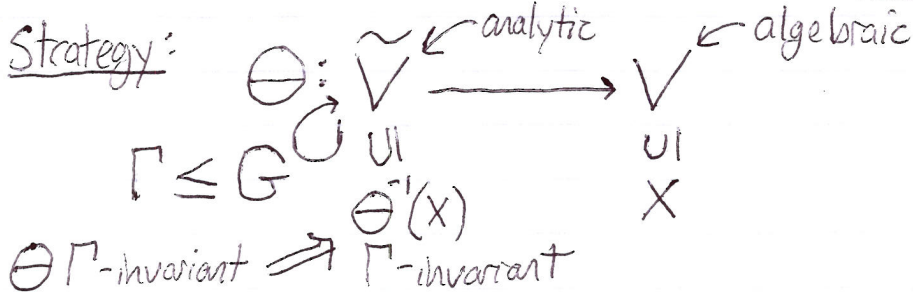
Example:

$\{(x, y, z) \mid y = x^z\}$ is definable in \mathbb{R}_{exp}

The fibers over rational z are semi-algebraic, so this set has a dense family of one-dimensional semi-algebraic subsets, but no two-dimensional semi-algebraic subset.

The "algebraic" part of this set is a very non-definable object.

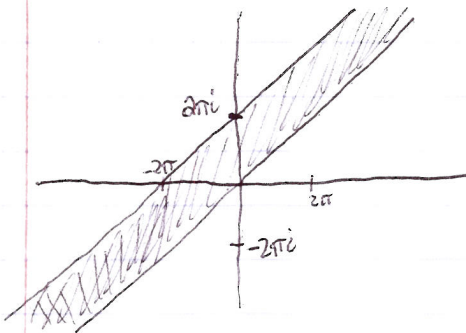
Strategy:



curve (a priori very tiny)
from Pila-Wilkie.
take its Zariski closure,

Notation: $\text{Stab}_G(\tilde{A}) = \{g \in G \mid g\tilde{A} = \tilde{A}\}$

Alternative fundamental set for the exponential case:



But the complex exponential map is not definable on this set in an o-minimal structure!
 If the imaginary part of a set is not bounded, defining exp on it will contradict o-minimality.

O-minimality and Arithmetic. The Pila-Zannier method

Kobi Peterzil

Department of Mathematics
University of Haifa

MSRI workshop, February 2014

Some Bibliography

J. Pila and U. Zannier, *Rational points in periodic analytic sets and the Manin-Mumford conjecture*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 19 (2008), no. 2, 149-162.

J. Pila, O-minimality and the André–Oort conjecture for \mathbb{C}^n . Ann. of Math. (2), 172(3), 2011, 1779–1840.

Survey papers

T. Scanlon, *A proof of the André–Oort conjecture via mathematical logic [after Pila, Wilkie and Zannier]*, Séminaire BOURBAKI Avril 2011 63ème année, 2010–2011, no 1037.

T. Scanlon, *Counting special points: Logic, diophantine geometry, and transcendence theory*, Bull. AMS (N.S.) 49 (2012), no. 1, 51 – 71.

A general problem scheme

Setting

\mathcal{C} = an underlying family of sets

$\mathcal{S} \subseteq \mathcal{C}$ is a collection of so-called “special” \mathcal{C} -sets

\mathcal{S}_0 = a set of so-called “special” points, often these are the \mathcal{S} -sets of dimension zero.

The problem scheme

Start with an \mathcal{S} -set V and consider an arbitrary \mathcal{C} -set $X \subseteq V$. Assume that X has “many” special points. Then X contains a special set of positive dimension. Under additional assumptions, X itself is a special set.

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The Pila-Wilkie results (viewed in this scheme)

Fix $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, \dots \rangle$ an o-minimal expansion of the real field.

\mathcal{C} = the family of all definable sets in \mathcal{M} .

\mathcal{S} = The family of semi-algebraic sets (defined over \mathbb{Q}).

\mathcal{S}_0 = points in $(\mathbb{Q}^{alg})^n \cap \mathbb{R}^n$.

The Pila-Wilkie theorem(s)

Assume that $X \subseteq \mathbb{R}^n$ is definable in \mathcal{M} . If $X \cap (\mathbb{Q}^{alg})^n$ is *large** then X contains a connected infinite semi-algebraic set defined over \mathbb{Q} .

More precisely, if one removes **all** infinite connected semi-algebraic subsets of X then a *small** number of \mathbb{Q}^{alg} -points remains.

$X \cap (\mathbb{Q}^{alg})^n$ is *large** if exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that

$$\limsup_T \frac{|\{\bar{q} \in X \cap (\mathbb{Q}_k^{alg})^n : \text{height}_k(\bar{q}) \leq T\}|}{T^\epsilon} = \infty.$$

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From now on-the algebraic general problem scheme

The algebraic presentation

\mathcal{C} = complex algebraic (irreducible) varieties, (quasi) affine or projective.

\mathcal{S} = a specified subfamily of “special” varieties.

\mathcal{S}_0 = 0-dimensional \mathcal{S} -sets: special points.

V = an irreducible \mathcal{S} -variety.

$X \subseteq V$ = an irreducible complex algebraic subvariety

Assumption

The set $X \cap \mathcal{S}_0$ is Zariski dense in X .

Goal

The variety X is itself in \mathcal{S} .

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A test case-the exponential example (algebraic torus)

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Let $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$ (so here V admits the structure of an algebraic group, which is also a complex Lie group).

$\mathcal{C} = \{X \subseteq (\mathbb{G}_m)^n : X \text{ an irreducible algebraic variety}\}$.

$\mathcal{S} = \{A + \rho : A \text{ a conn. algebraic subgrp of } \mathbb{G}_m^n \text{ \& } \rho \text{ a torsion point}\}$.

$\mathcal{S}_0 = \text{Torsion points in } (\mathbb{G}_m)^n$

Theorem (Laurent)

If $X \subseteq (\mathbb{G}_m)^n$ an irreducible algebraic variety and $X \cap \text{Tor}(\mathbb{G}_m)^n$ is Zariski dense in X then $X = A + \rho$ for some $A \leq (\mathbb{G}_m)^n$ and $\rho \in \text{Tor}(\mathbb{G}_m)^n$.

Namely,

if $X \in \mathcal{C}$ and $X \cap \mathcal{S}_0$ is Zariski dense in X then $X \in \mathcal{S}$.

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Back to the general problem-the analytic presentation

We work with affine (or projective) algebraic variety V and an algebraic subvariety $X \subseteq V$.

An analytic covering map

We have \tilde{V} = a (semi-algebraic) open subset of \mathbb{C}^n (with $n = \dim V$).
And $\Theta : \tilde{V} \rightarrow V$ a holomorphic, **transcendental**, surjection.

General strategy

Replace V and its algebraic variety $X \subseteq V$ by \tilde{V} and a complex analytic subvariety $\Theta^{-1}(X) \subseteq \tilde{V}$.

Caution

In general, Θ and $\Theta^{-1}(X)$ are not definable in any “tame” structure.
We will need to “truncate” them.

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In general, Θ and $\Theta^{-1}(X)$ are not definable in any “tame” structure.
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An underlying group action

We have $G =$ a real algebraic group acting semi-algebraically and transitively on \tilde{V} . In some cases $\tilde{V} = G$.

$\Gamma =$ an infinite discrete subgroup of G (not necessarily normal).

The map $\Theta : \tilde{V} \rightarrow V$ is Γ -invariant. Namely, $\Theta(x) = \Theta(y)$ if and only if $\Gamma x = \Gamma y$.

So, V can be identified with $\Gamma \backslash \tilde{V}$.

If $X \subseteq V$ is a complex algebraic subvariety then $\Theta^{-1}(X) = \tilde{X}$ is a Γ -invariant analytic subvariety of \tilde{V} .

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$\widetilde{\text{special}}$ -varieties and points

An irreducible analytic subvariety $Y \subseteq \widetilde{V}$ is called a $\widetilde{\text{special}}$ variety if $\Theta(Y)$ is a spacial subvariety of V . In particular, $\Theta(Y)$ is algebraic (!).

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Recall $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$. Take $\tilde{V} = \mathbb{C}^n$ and $\Theta := \exp : \mathbb{C}^n \rightarrow (\mathbb{C}^*)^n$ defined by $\exp(z_1, \dots, z_n) = (e^{z_1}, \dots, e^{z_n})$.

So $\Theta : (\mathbb{C}^n, +) \rightarrow ((\mathbb{G}_m)^n, *)$ is a holomorphic group homomorphism.

$\Gamma := \text{Ker}(\Theta) = (2\pi i\mathbb{Z})^n$. Clearly, Θ is Γ -invariant.

special points

Because Θ is a homomorphism, $\Theta(z)$ is a torsion point of order k iff $kz \in \Gamma$. So, $\tilde{S}_0 = \{\bar{z} \in \mathbb{C}^n : \exists k \ k\bar{z} \in (2\pi i\mathbb{Z})^n\} = (2\pi i\mathbb{Q})^n$.

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An irreducible analytic $Y \subseteq \mathbb{C}^n$ is special if $\Theta(Y) = p + A$, where A is an algebraic subgroup of $(\mathbb{G}_m)^n$ and $p \in \text{Tor}(\mathbb{G}_m)^n$. So, $Y = \bar{q} + H$, where H is a \mathbb{C} -linear subspace of \mathbb{C}^n **defined over** \mathbb{Q} , and $\bar{q} \in (2\pi i\mathbb{Q})^n$. (Note: these are exactly the cosets of real subgroups of \mathbb{C}^n which project onto algebraic varieties).

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So $\Theta : (\mathbb{C}^n, +) \rightarrow ((\mathbb{G}_m)^n, *)$ is a holomorphic group homomorphism.

$\Gamma := \text{Ker}(\Theta) = (2\pi i\mathbb{Z})^n$. Clearly, Θ is Γ -invariant.

special points

Because Θ is a homomorphism, $\Theta(z)$ is a torsion point of order k iff $kz \in \Gamma$. So, $\tilde{S}_0 = \{\bar{z} \in \mathbb{C}^n : \exists k \ k\bar{z} \in (2\pi i\mathbb{Z})^n\} = (2\pi i\mathbb{Q})^n$.

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An irreducible analytic $Y \subseteq \mathbb{C}^n$ is special if $\Theta(Y) = p + A$, where A is an algebraic subgroup of $(\mathbb{G}_m)^n$ and $p \in \text{Tor}(\mathbb{G}_m)^n$. So, $Y = \bar{q} + H$, where H is a \mathbb{C} -linear subspace of \mathbb{C}^n defined over \mathbb{Q} , and $\bar{q} \in (2\pi i\mathbb{Q})^n$.

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The general case: The (non) definability of Θ

We have $\Theta : \tilde{V} \rightarrow V \sim \Gamma \backslash \tilde{V}$

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Since Γ is infinite discrete, $\Theta^{-1}(p)$ is an infinite discrete set (for every $p \in V$). Hence, the map Θ is never definable in an o-minimal structure.

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The ingredients for the Pila-Zannier method

We have $\Theta : \tilde{V} \rightarrow V \sim \Gamma \backslash \tilde{V}$. $S_0 \subseteq V$ the set of special points.

I. Definability requirements (from algebraic to o-minimal)

One needs to establish the existence of a semialgebraic fundamental set $\mathfrak{F} \subseteq \tilde{V}$ and the definability of $\Theta \upharpoonright \mathfrak{F}$ in some o-minimal structure \mathcal{M} . In all examples, \mathcal{M} is $\mathbb{R}_{an,exp}$.

For $X \subseteq V$ algebraic, let $\tilde{X} \subseteq \tilde{V}$ be an irreducible analytic component of $\Theta^{-1}(X)$. Note that $\tilde{X} \cap \mathfrak{F} = (\Theta \upharpoonright \mathfrak{F})^{-1}(X)$ is definable in \mathcal{M} .

II. Number theory goal

- The set $\tilde{S}_0 = \Theta^{-1}(S_0)$ is contained in \mathbb{Q}_k^{alg} for some k (up to definable bijection).
- ▶▶ If $X \cap S_0$ (on the algebraic side) is Zariski dense in X then $\tilde{S}_0 \cap (\tilde{X} \cap \mathfrak{F})$ (on the analytic side) is large* (in the sense of Pila-Wilkie). This is “the lower bound”.

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The Pila-Wilkie input

- Assume that we established that $\tilde{S}_0 \cap (\tilde{X} \cap \mathfrak{F})$ is large*.
- By PW, There exists a connected semi-algebraic nontrivial curve $C \subseteq \tilde{X} \cap \mathfrak{F}$.
- Let $\overline{C} \subseteq \mathbb{C}^n$ be the Zariski closure of C . It is a complex algebraic curve, and by dimension considerations $(\overline{C} \cap \tilde{V}) \subseteq \tilde{X}$.
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The general idea

Take \tilde{A} a maximal algebraic subset of $\Theta^{-1}(X)$.

The Γ -periodicity of $\Theta^{-1}(X)$ together with the algebraicity of \tilde{A} is “unlikely” and should imply that the stabilizer of \tilde{A} in $G(\mathbb{R})$ is nontrivial. In fact, it should imply that \tilde{A} is “special”.

More precisely,

Ingredient III, the “Ax-Lindemann” goal

Assume that \tilde{A} is a maximal irreducible algebraic (relative to \tilde{V}) subset of \tilde{X} .

Then \tilde{A} is a weakly special variety. Namely,

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We have $\Theta : \tilde{V} \rightarrow V \sim \Gamma \backslash \tilde{V}$. $\tilde{X} \subseteq \tilde{V}$ a component of $\Theta^{-1}(X)$.

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Take \tilde{A} a maximal algebraic subset of $\Theta^{-1}(X)$.

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More precisely,

Ingredient III, the “Ax-Lindemann” goal

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Summary of the Pila-Zannier method

We have $X \subseteq V$, $\Theta : \tilde{V} \rightarrow V$ and $X \cap \mathcal{S}_0$ Zariski dense in X .

I. Definability

$\Theta \upharpoonright \mathfrak{F}$ is definable in an o-minimal structure.

II. Number Theory

The set $\tilde{\mathcal{S}}_0 \cap (\Theta^{-1}(X) \cap \mathfrak{F})$ is large*.

Application of the Pila-Wilkie Theorem.

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If $\tilde{A} \subseteq \Theta^{-1}(X)$ is maximal irreducible algebraic then it is weakly special. (So, in addition $\tilde{A} \cap \tilde{\mathcal{S}}_0 \neq \emptyset$ then \tilde{A} is special).

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Applying Pila-Zannier to the exponential case

We have $V = (\mathbb{C}^*)^n$

$\mathcal{S}_0 = \text{Tor}(\mathbb{C}^*)^n$

$\mathcal{S} = \{A + p : A \in (\mathbb{C}^*)^n, p \in \mathcal{S}_0\}$.

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We have $\Theta : \mathbb{C}^n \rightarrow (\mathbb{C}^*)^n$ given by $\Theta(z_1, \dots, z_n) = (e^{z_1}, \dots, e^{z_n})$.

• A fundamental set for Θ is:

$$\mathfrak{F} = \{\bar{z} = (z_1, \dots, z_n) \in \mathbb{C}^n : 0 \leq |\text{Im}(z_j)| \leq \pi\}.$$

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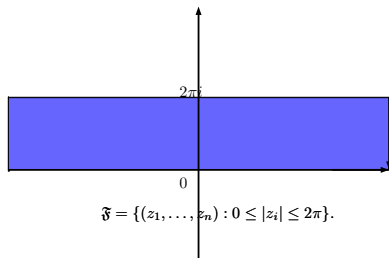
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$\Theta \upharpoonright \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$:

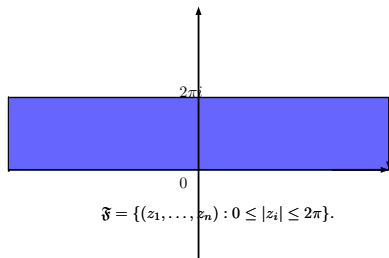
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The map e^x is definable in \mathbb{R}_{exp} ; the maps $\cos, \sin \upharpoonright [0, 2\pi]$ are definable in \mathbb{R}_{an} , hence:

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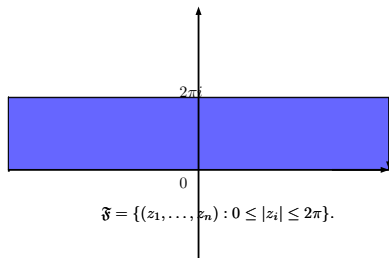
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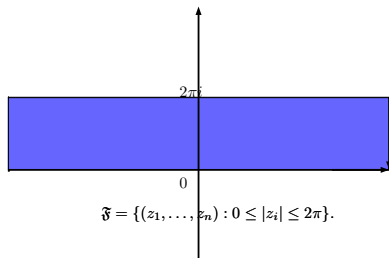
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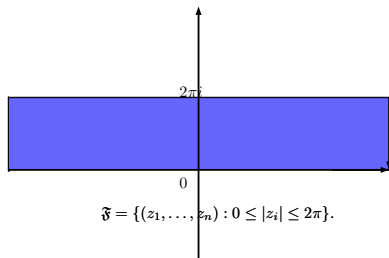
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We want to show that $(2\pi i\mathbb{Q})^n \cap (\Theta^{-1}(X) \cap \mathfrak{F})$ is large*.

- X is defined over a number field k . For simplicity, $k = \mathbb{Q}$.
- Since $X \cap \text{Tor}(\mathbb{C}^*)^n$ is infinite there are natural numbers $m_1 < m_2 < \dots$ and elements $g_i \in X$, with $\text{ord}(g_i) = m_i$.
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The Pila-Wilkie input

The analytic set $\Theta^{-1}(X) \subseteq \mathbb{C}^n$ contains a nontrivial algebraic set. Take a maximal such irreducible algebraic set A .

Goal: A is weakly special = a coset of a linear s.space of \mathbb{C}^n over \mathbb{Q} .

A proof using the classical Ax's theorem (corrected)

Ax's Theorem If $\xi_1, \dots, \xi_n \in \mathbb{C}(A)$ and $\text{lin. dim}_{\mathbb{Q}}(\bar{\xi}/\mathbb{C}) = m$ then $\text{tr. deg}(\mathbb{C}(\exp(\xi_1), \dots, \exp(\xi_n))/\mathbb{C}) = m$.

- Take $H \subseteq \mathbb{C}^n$ a minimal subspace $/\mathbb{Q}$ with $A \subseteq H + p$ for $p \in \mathbb{C}^n$. Let $m = \dim H$.
- We have $\Theta(A) \subseteq \Theta(H) + \Theta(p)$, and $\Theta(H) \leq (\mathbb{C}^*)^n$ algebraic.
- Take $\xi_1, \dots, \xi_n \in \mathbb{C}(A)$ coordinate functions in the function field of A . Then $\text{lin. dim}_{\mathbb{Q}}(\bar{\xi}/\mathbb{C}) = m$, so by Ax's theorem $\text{tr. deg}(\mathbb{C}(\Theta(\bar{\xi}))/\mathbb{C}) = m = \dim(\Theta(H) + \Theta(p))$.

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It follows that $\Theta(A)$ is Zariski dense in $\Theta(H) + \Theta(p)$, so $\Theta(H) + \Theta(p) \subseteq X$.

Hence, $A \subseteq H + p \subseteq \Theta^{-1}(X)$.

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- We started with $X \subseteq (\mathbb{G}^m)^n$ such that $Tor(\mathbb{G}_m)^n \cap X$ is Zariski dense in X .
- Using Pila-Wilkie, we concluded that $\Theta^{-1}(X)$ contained a nontrivial complex algebraic set A . Furthermore we can choose it so $A \cap \tilde{\mathcal{S}}_0$ is nonempty. Take such A maximal.
- By Ax, A is weakly $\widetilde{\text{special}}$, hence $\widetilde{\text{special}}$ ($A \cap \tilde{\mathcal{S}}_0 \neq \emptyset$).
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Another example: The Manin-Mumford conjecture

The setting

V = an abelian variety in $\mathbb{P}^n(\mathbb{C})$.

So, V is a projective algebraic variety which admits an algebraic group structure, abelian. It is also a compact, complex Lie group.

\mathcal{C} = all irreducible algebraic subvarieties of V .

\mathcal{S} = all cosets of the form $A + p$, where $p \in \text{Tor}(V)$ and A a connected algebraic subgroup (i.e. abelian subvariety) of V .

$\mathcal{S}_0 = \text{Tor}(V)$ the torsion elements.

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Assume that V is a complex abelian variety defined over a number field, and $X \subseteq V$ an irreducible algebraic subvariety. If $X \cap \text{Tor}(V)$ is Zariski dense in V then $X = A + p$ as above.

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The analytic presentation

- There exists a holomorphic group homomorphism $\Theta : (\mathbb{C}^n, +) \rightarrow V$.
- $\Gamma := \text{Ker}(\Theta)$ is a $2n$ -lattice. I.e., $\Gamma = \sum_{i=1}^{2n} \mathbb{Z}\omega_i$, where $\omega_1, \dots, \omega_{2n}$ are linearly independent over \mathbb{R} .
(Note: While every $2n$ -lattice gives rise to a complex torus, it might not give rise, if $n > 1$, to an **projective** complex torus, i.e. abelian variety.)
- **special points** $= \Theta^{-1}(\text{Tor}(V)) = \mathbb{Q}\Gamma = \sum_{i=1}^{2n} \mathbb{Q}\omega_i$.
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(weakly) special varieties as orbits

Note that the weakly special varieties are exactly those orbits (i.e., cosets) of real subgroups of $(\mathbb{C}^n, +)$ which project onto algebraic subvarieties of V .

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The Pila-Zannier method for Manin-Mumford

I. The fundamental set and definability of $\Theta \upharpoonright \mathfrak{F}$

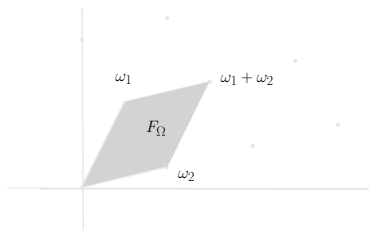
Consider the compact semilinear parallelogram

$\mathfrak{F} = \{ \sum_{i=1}^{2n} t_i \omega_i : 0 \leq t_i \leq 1 \}$. Then:

(i) $\Gamma + \mathfrak{F} = \mathbb{C}^n$.

(ii) The set $\{ \gamma \in \Gamma : (\gamma + \mathfrak{F}) \cap \mathfrak{F} \neq \emptyset \}$ is finite.

\mathfrak{F} is a fundamental set for Θ .



Since Θ is analytic on \mathbb{C}^n and \mathfrak{F} compact, $\Theta \upharpoonright \mathfrak{F}$ is definable in the o-minimal \mathbb{R}_{an} (by considering the real and imaginary parts of Θ).

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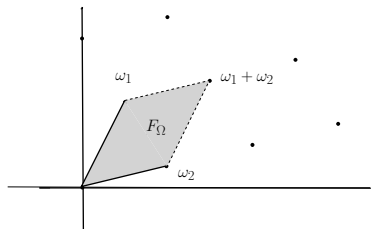
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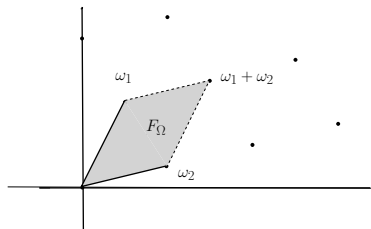
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Since Θ is analytic on \mathbb{C}^n and \mathfrak{F} compact, $\Theta \upharpoonright \mathfrak{F}$ is definable in the o-minimal \mathbb{R}_{an} (by considering the real and imaginary parts of Θ).

The Pila-Zannier method for Manin-Mumford

I. The fundamental set and definability of $\Theta \upharpoonright \mathfrak{F}$

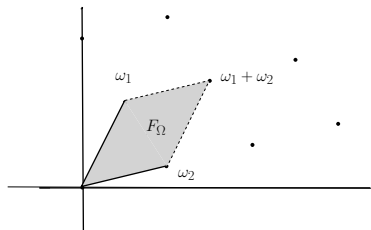
Consider the compact semilinear parallelogram

$\mathfrak{F} = \{ \sum_{i=1}^{2n} t_i \omega_i : 0 \leq t_i \leq 1 \}$. Then:

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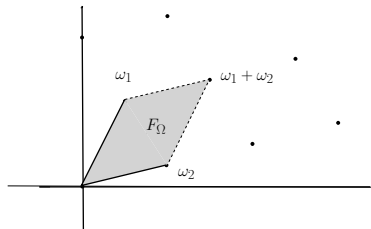
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II. Number Theory (on the algebraic side)

- V is an abelian variety defined over a number field F .
- $X \subseteq V$ is irreducible algebraic, with $X \cap \text{Tor}(V)$ Zariski dense in X .
- So, X is also defined over a number field $k \supseteq F$.

Number theoretic input (Masser)

There exists $\rho = \rho(V) > 0$ and a constant c , such that for every $P \in V$, if $\text{ord}(P) = T$ then $[F(P) : \mathbb{Q}] \geq cT^\rho$.

By conjugating $X \cap \text{Tor}(V)$ over k we conclude: if $\epsilon < \rho(V)$ then

$$\limsup_T \frac{|\{P \in X : \text{ord}(P) \leq T\}|}{T^\epsilon} = \infty.$$

Conclusion: on the analytic side

The set $\{(q_1, \dots, q_{2n}) \in \mathbb{Q}^{2n} : \sum_{j=1}^{2n} q_j \omega_j \in \Theta^{-1}(X) \cap \mathfrak{F}\}$ is large*.

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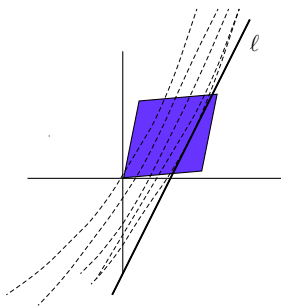
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III. Ax-Lindemann: an o-minimal argument

The Pila-Wilkie input

The analytic variety $\Theta^{-1}(X)$ contains an unbounded semialgebraic curve σ .

By the o-minimality of σ , when we translate it into \mathfrak{F} by elements of Γ we get (inside \tilde{X}) curves which are more and more “linear”. Since $\tilde{X} \cap \mathfrak{F}$ is compact, at the limit we get an affine line $\ell \subseteq \tilde{X}$.

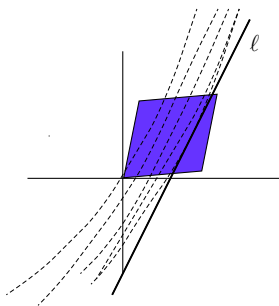


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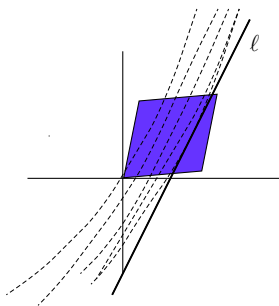


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Finishing the proof of MM

On the analytic side

We saw that $\Theta^{-1}(X)$ contains an affine line $\ell \subseteq \mathbb{C}^n$.

Back to the algebraic side

The variety $X \subseteq V$ contains $\Theta(\ell)$, a coset of a subgroup.

The Zariski closure of $\Theta(\ell)$ is a coset of an algebraic subgroup of V , which is contained in X .

Hence, X contains a (weakly) special variety $z + A$, for $A \leq X$.

By using the full strength of Pila-Wilkie, together with the ability to write V as an almost direct product $A \oplus B$, we can show that X itself is a special variety.

END of the proof of Manin-Mumford.

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The general analytic setting for Shimura varieties (simplified)

- $G(\mathbb{R})$ is the \mathbb{R} -points of an algebraic semisimple group G over \mathbb{R} .
- $K \leq G(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$.
- (with additional assumptions) the quotient space $G(\mathbb{R})/K$ admits the structure of an open semi-algebraic subset of \mathbb{C}^n . This set is our \tilde{V} .
- $G(\mathbb{R})$ acts on \tilde{V} . Actually, for every $g \in G(\mathbb{R})$, $g : \tilde{V} \rightarrow \tilde{V}$ is a biholomorphism.
- Let $\Gamma = G(\mathbb{Z})$ (more generally, an arithmetic subgroup), and consider the quotient $V = \Gamma \backslash \tilde{V}$.

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There exists a holomorphic embedding $\Theta : \Gamma \backslash \tilde{V} \rightarrow \mathbb{P}^m(\mathbb{C})$ whose image is a quasi-projective variety.

$Im(\Theta) = V$ is a *Shimura variety* (a non-specialist viewpoint).

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The Shimura variety \mathbb{C}^n : Preliminaries

We start with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$.

The group $SL(2, \mathbb{R})$ acts on \mathbb{H} (transitively) as follows:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\tau \in \mathbb{H}$ then $A \cdot \tau = \frac{a\tau + b}{c\tau + d}$.

Connection to elliptic curves

\mathbb{H} is a parameter space for elliptic curves, namely, every τ represents the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ where Λ_τ the lattice $\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau$.

$E_{\tau_1} \cong E_{\tau_2} \Leftrightarrow \tau_1, \tau_2$ are in the same $SL(2, \mathbb{Z})$ -orbit. So, $SL(2, \mathbb{Z}) \backslash \mathbb{H}$ is the moduli space of elliptic curves.

The J -invariant

There exists a holomorphic, transcendental surjection $J : \mathbb{H} \rightarrow \mathbb{C}$ such that $J(\tau_1) = J(\tau_2) \Leftrightarrow SL(2, \mathbb{Z})\tau_1 = SL(2, \mathbb{Z})\tau_2$.

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Connection to elliptic curves

\mathbb{H} is a parameter space for elliptic curves, namely, every τ represents the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ where Λ_τ the lattice $\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau$.

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Special varieties and points

Again, the definition begins on the analytic side.

Definition of $\widetilde{\mathfrak{S}}_0$: The set $\widetilde{\mathfrak{S}}_0$

$(\tau_1, \dots, \tau_n) \in \mathbb{H}^n$ is **special**, if for every i , the elliptic curve E_{τ_i} has complex multiplication ($\text{End}(E_{\tau_i}) \neq \mathbb{Z}$).

Equivalently, τ_i belongs to an imaginary quadratic extension of \mathbb{Q} .

(abstract definition of $\widetilde{\mathfrak{S}}_0$ in Shimura varieties-omitted here).

Definition of special varieties

Recall: An irreducible analytic variety $Y \subseteq \mathbb{H}^n$ is **special** if

- (i) Y is an orbit of a real algebraic group $H \leq \text{SL}(2, \mathbb{R})^n$.
- (ii) $\theta(Y) \subseteq \mathbb{C}^n$ is an algebraic variety.
- (iii) $Y \cap \widetilde{\mathfrak{S}}_0 \neq \emptyset$.

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Special varieties and points in $V = \mathbb{C}^n$

The image under Θ of a special point is **special** in \mathbb{C}^n . $s_0 := \Theta(\tilde{s}_0)$.

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Examples of special varieties

• $\tilde{X} = \{\tau\} \times \mathbb{H}^{n-1}$, with $\tau \in \tilde{s}_0$; it is an orbit of $H = \{1\} \times SL(2, \mathbb{R})^{n-1}$.

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Moonen's work

Every special variety in \mathbb{C}^n is obtained from the above examples by permutation of variables and cartesian products.

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The statement of the theorem

The André-Oort Conjecture for \mathbb{C}^n (a theorem of Pila)

If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety and $X \cap \mathcal{S}_0$ is Zariski dense in X then X is special.

By the nature of the definitions, we immediately have an analytic presentation of the problem:

- We have $\Theta : \mathbb{H}^n \rightarrow \mathbb{C}^n$ given by the J function in each coordinate.
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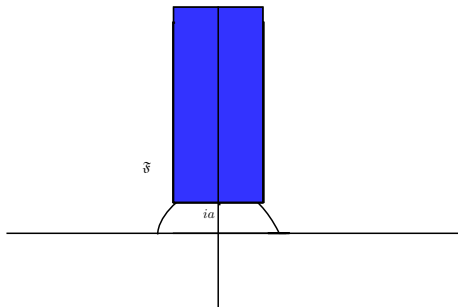
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The Pila Zannier method: I. The fundamental set

By the basic theory of elliptic curves, the following is a fundamental set for $SL(2, \mathbb{Z})$ (for every $0 < a < \sqrt{3}/2$):

$$\mathfrak{F} = \{z \in \mathbb{H} : -1/2 \leq \operatorname{Re}(z) \leq 1/2 \text{ \& } \operatorname{Im}(z) > a\}.$$



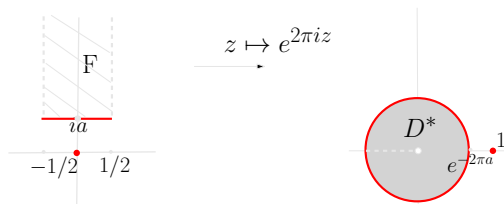
So \mathfrak{F}^n is a fundamental set for $SL(2, \mathbb{Z})^n$.

Pila-zanner method I: Definability of $J \upharpoonright \mathfrak{F}$

Theorem

The restriction of J to \mathfrak{F} is definable in $\mathbb{R}_{an,exp}$.

Proof Consider first the map $z \mapsto e^{2\pi iz}$. It sends \mathfrak{F} onto a punctured disc D^* . The “point” $Im(z) = \infty$ is sent to 0 .



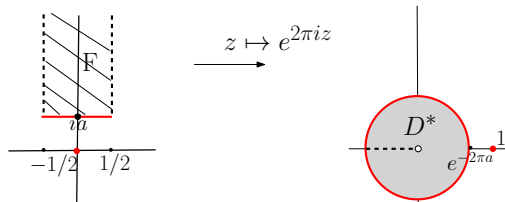
- The restriction of $e^{2\pi iz}$ to \mathfrak{F} is definable in $\mathbb{R}_{an,exp}$.
- As pointed out in an earlier talk, we may write J in the variable $q = e^{2\pi iz}$ and obtain a meromorphic function on D^* . Hence (???) $J(q)$ is definable in \mathbb{R}_{an} . It follows that $J(z)$ is definable in $\mathbb{R}_{an,exp}$.

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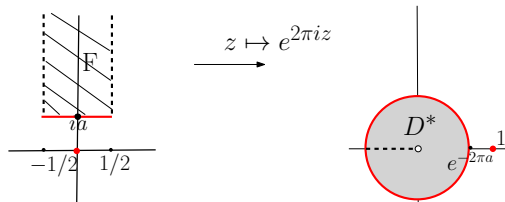
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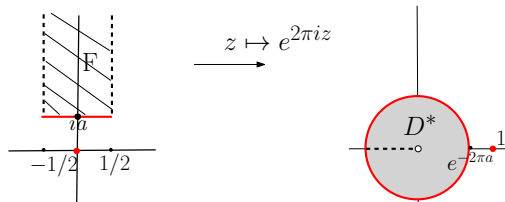
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II. Number Theory

We have $\Theta : \mathbb{H}^n \rightarrow \mathbb{C}^n$, and $X \subseteq \mathbb{C}^n$ algebraic, with $X \cap \mathcal{S}_0$ Zariski dense in X . We use \mathfrak{F} for the fundamental set for $\Theta (= \mathfrak{F}^n)$.

On the analytic side

Let $\tilde{X} \subseteq \mathbb{H}^n$ be an irreducible **analytic** component of $\Theta^{-1}(X)$.

We already saw that if $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{H}^n$ is special then each τ_i is imaginary quadratic, so $\tilde{\mathcal{S}}_0 \subseteq (\mathbb{Q}_2^{alg})^n$.

Using a theorem of Siegel on imaginary quadratic fields, Pila proves:

Largeness of special points

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III. The Ax-Lindemann statement

The Pila-Wilkie input

\tilde{X} contains an algebraic set of positive dimension (relative to \mathbb{H}^n). Let A be maximal irreducible such set.

Goal

A is weakly special. Namely

- (i) it is the orbit of a real algebraic subgroup of $SL(2, \mathbb{R})^n$, and
- (ii) $\Theta(A)$ is algebraic.

Ax-Lindemann for \mathbb{H}^n (third type of proof)

- We have $A \subseteq \Theta^{-1}(X)$ a maximal, irreducible relatively algebraic subset, of positive dimension. Namely, there exists an algebraic $\bar{A} \subseteq \mathbb{C}^n$ such that $A = \bar{A} \cap \mathbb{H}^n$.

Write $G := SL(2, \mathbb{R})^n$, and $\Gamma = SL(2, \mathbb{Z})^n$.

- Without loss of generality $\dim(A \cap \mathfrak{F}) = \dim A$ (if not, replace \tilde{X} and A by $\gamma\tilde{X}$ and γA , for some $\gamma \in \Gamma$).

Fact A is not contained in finitely many Γ -translates of \mathfrak{F} .

WHY?

Otherwise $A \subseteq \bigcup_{i=1}^k \gamma_i \mathfrak{F}$. Because the real part of \mathfrak{F} is bounded, it follows that $\operatorname{Re}(z)$ is bounded for $z \in \bar{A} \cap \mathbb{H}^n$. This would imply (?) that A must be compact. But a compact complex analytic subset of \mathbb{H}^n is finite. Contradiction.

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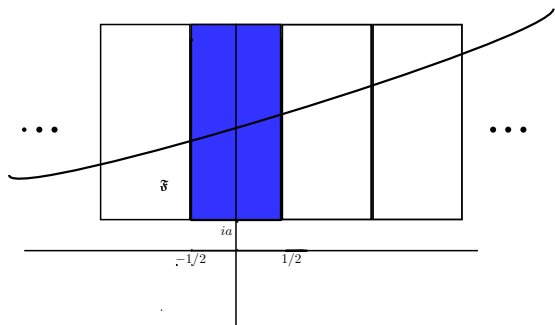
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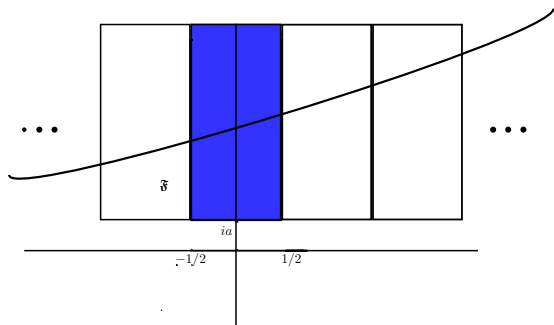
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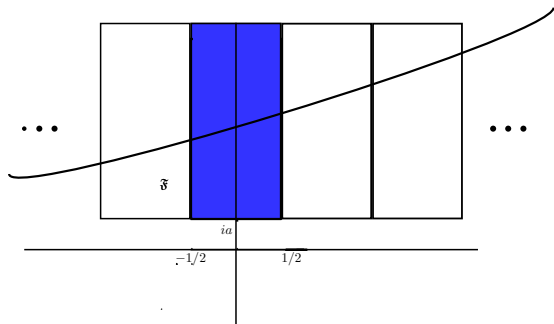
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The set $\{\gamma \in SL(2, \mathbb{Z})^n : \gamma \in G(A)\}$ is large*.

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Theorem The André- Oort conjecture holds for \mathcal{A}_2 , the moduli space of abelian surfaces.

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Recent work of Klingler, Yafaev and Ullmo (2013)

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