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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Lou van der Dries

Talk Title: Model theory and multiplicative combinatorics (II)

Date: 02 / 06 / 14 **Time:** 2 : 30 am / pm (circle one)

List 6-12 key words for the talk: groups of polynomial growth, Breuillard-Green-Tao, approximate subgroups, Margulis Lemma

Please summarize the lecture in 5 or fewer sentences: Part 2 of 3. A closer analysis of the weakened BGT theorem proof from Part 1, focusing on the source of the $3\log_2 K$ bound. An application to a variation and a strong finitary form of Gromov's theorem on groups of polynomial growth. A discussion of the generalized Margulis Lemma at the end of the talk.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
    - **Overhead**: Obtain a copy or use the originals and scan them
    - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Plan for today

1° Recall BGT-theorem

2° Indicate where $3\log_2 K$-bound comes from

3° Applications of BGT: improving Gromov's thm on groups of polynomial growth, generalized Margulis Lemma

Plan for tomorrow

Back to limits of approximate groups: how to get a locally compact group from a pseudofinite K-approximate group $X \subseteq G$. This involves some useful model-theoretic generalities

BGT-theorem

If $X \subseteq G$ is a finite K-approximate group, then there is a K^6-approximate group $Y \subseteq X^4$, s.t.

(i) $X \subseteq EY$, $|E| \leq L = L(K)$

(ii) $\langle Y \rangle$ is $\underbrace{\text{virtually } d\text{-nilpotent}}_{\text{has } d\text{-nilpotent subgroup of finite index}}$, $d \leq 3 \log_2 K$

This is weaker than the version I stated last time, but enough for the applications to be discussed. Stronger versions are needed for other applications, to Cayley graphs of finite groups, see

Benjamini, Finucane, Tessera "On the scaling limit of finite vertex transitive graphs with large diameter"

$3 \log_2 K$ - bound (Hrushovski):

In the limit setting of the proof of BGT we have a group morphism $\rho : \langle Y \rangle \to \mathcal{H}$ onto a connected Lie group $\mathcal{H}$. The proof shows that $\mathcal{H}$ is <u>nilpotent</u>. Dividing out by the largest compact subgroup of $\mathcal{H}$ we arrange that $\mathcal{H}$ is homeomorphic to $\mathbb{R}^d$, $d = \dim \mathcal{H}$. Moreover, $\mathcal{H}$ has a compact neighborhood C of the identity with $\mu(C^2) \leq K^3 \mu(C)$. Also:

$$\mu(C^2) \geq 2^d \mu(C)$$

$$\therefore \quad d \leq 3 \log_2 K$$

Gromov: Let $G = \langle S \rangle$, $S \subseteq G$ finite and symmetric, such that $|S^n| \leq c n^d$ for all n (fixed $c, d \in \mathbb{N}$). Then G is virtually nilpotent

"for all n" can be replaced by "for infinitely many n"

Gromov's proof uses the solution of H5

A few years ago Kleiner found a proof

not using H5

Hrushovski: if G is finitely generated and $K \geq 1$ is such that $G = \bigcup_n X_n$ with $X_n \subseteq G$ finite symmetric and $|X_n|^2 \leq K|X_n|$ for all n, then G is virtually nilpotent

Let's see how a finitary form of Gromov's theorem can be obtained from BGT

Lemma Let $G = \langle S \rangle$, $\underset{\text{symmetric}}{S \subseteq} G \geq G_1$

If $[G : G_1] = d$, then S^n meets at least $n+1$ left cosets of G_1, for all $n < d$.

Pf: clear for $n = 0$. Assume towards contradiction that $0 < n < d$ and S^n meets at most n left cosets of G_1. Considering $S^0, S^1, \ldots, S^n$ and using PHP, get $i < n$ such that S^i and S^{i+1} meet the same left cosets of G_1.

$\therefore \quad S^i G_1 = S^{i+1} G_1 = S^{i+2} G_1 = \cdots$

$\therefore \quad S^i G_1 = \langle S \rangle G_1 = G$, so S^i meets d left cosets of G_1, contradiction. $\boxtimes$

<u>Cor</u> Let $K \geq 1$ be given, let $L \in \mathbb{N}^{\geq 1}$ be as in BGT. Suppose $X \subseteq G$ is a finite K-appr. group and $S \subseteq G$ is symmetric, $S^L \subseteq X$. Then

$\langle S \rangle$ is virtually d-nilpotent, $d \leq 3 \log_2 K$.

<u>Pf</u> Take $Y \subseteq X^4$ as in BGT, so X is covered by L left translates of Y.

$\therefore$ $S^L \subseteq X$ is covered by L left cosets of $G_1 := \langle Y \rangle \cap \langle S \rangle$

$\therefore [\langle S \rangle : G_1] \leq L$

by previous lemma

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
\langle Y \rangle \arrow[dr, dash] & & \langle S \rangle \arrow[dl, dash, "\leq L"] \\
 & G_1 &
\end{tikzcd}$$

BGT: $\langle Y \rangle$ is virtually d-nilpotent, $d \leq 3 \log_2 K$

$\therefore$ G_1 is virtually d-nilpotent

$\therefore$ $\langle S \rangle$ is virtually d-nilpotent

<u>Easy</u> <u>Exercise</u> : if $c, d \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{R}^{>0}$ is increasing with $f(n) \le cn^d$ for all n, and $K > 5^d$, then $f(5n) \le K f(n)$ for infinitely many n

<u>Pf</u> of Gromov's <u>Theorem</u> : let $G = \langle S \rangle$ with finite symmetric $S \subseteq G$ such that $|S^n| \le cn^d$ for all n. Take $K > 5^d$. By "easy exercise" applied to $f(n) = |S^n|$, get $|S^{5n}| \le K|S^n|$ for infinitely many n. Let $L = L(K)$ be as in BGT. Take $n \geq L$ such that $|S^{5n}| \le K|S^n|$. Then $X := S^{2n}$ is a K-approximate group (lemma of last time) and $S^L \subseteq X$, so can apply result on <u>6</u> to conclude: $G = \langle S \rangle$ is virtually nilpotent

Finitary Gromov Thm: uses a finitization of the "easy exercise":

Lemma Let $d \in \mathbb{N}$, $K := 5^{d+1}$; $L \geq K$.

Let $f : \mathbb{N} \to \mathbb{R}^{>0}$ be increasing, $n \geq 5KL^{d+1}$.

Then: $f(n) \leq f(1)n^d \Rightarrow f(5m) \leq K f(m)$ for some m with $L \leq m \leq n/5$

The resulting finitary refinement of Gromov's theorem is of the kind that a degree d bound at just <u>one</u> big enough scale implies virtual nilpotence:

Finitary Gromov Theorem

Let $d \in \mathbb{N}$. Then there is $N(d) \in \mathbb{N}$ with the following property: if $G = \langle S \rangle$ with finite symmetric $S \subseteq G$ and $|S^n| \leq |S| n^d$ for *some* $n \geq N(d)$, then G is virtually $7(d+1)$-nilpotent

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With $K := 5^{d+1}$ and $L \geq K$, $L \in \mathbb{N}$ as in BGT, $N(d) := 5KL^{d+1}$ works.

Another application is the following

<u>Generalized Margulis Lemma</u> (conjectured by Gromov)

Let $K \in \mathbb{N}^{\geq 1}$ be given. Then there is an $\varepsilon = \varepsilon(K)$, $0 < \varepsilon \leq 2$, with the following property. Let (M,d) be a metric space and let a be a point in M such that the closed ball of radius 4 centered at a can be covered by K closed balls of radius 1.

Let Γ be a subgroup of $\mathrm{Isom}(M,d)$ such that $\{\gamma \in \Gamma : d(\gamma a, a) \leq 2\}$ is finite. Then the ε-stabilizer

$$S_\varepsilon(a) := \{\gamma \in \Gamma : d(\gamma a, a) \leq \varepsilon\}$$

generates a virtually d-nilpotent subgroup of Γ with $d \leq 3 \log_2 K$.

Assumptions in this Margulis Lemma are satisfied if (M,d) is a complete Riemannian manifold with a lower bound on its Ricci curvature, by Bishop-Gromov volume comparison estimates.

(In this case it was proved by Cheeger-Colding in '96.)