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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Terence Tao

Talk Title: A regularity lemma for definable sets over finite fields, and expanding polynomials

Date: 02/06/14 **Time:** 4:00 am (pm) (circle one)

List 6-12 key words for the talk: Szemeredi's regularity lemma, expanding polynomials, finite fields, Cauchy-Schwarz

Please summarize the lecture in 5 or fewer sentences: In the setting of definable relations between definable sets in finite fields, one can obtain stronger versions of Szemeredi's regularity lemma. This was proven by Tao, reproven by Pillay-Starchenko and Hrushovski in a more general setting, and then reproven by Tao with less model theory. The proof, and its application to expanding polynomials, were sketched.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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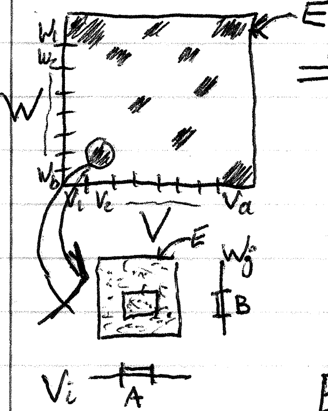
☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Szemerédi's regularity lemma

V, W finite sets, $E \subseteq V \times W$, $\varepsilon > 0$



$\Rightarrow \exists\ V = V_1 \cup \dots \cup V_a \quad a \le M(\varepsilon)$

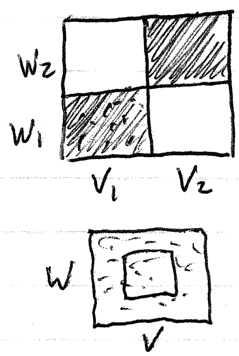
$W = W_1 \cup \dots \cup W_b \quad b \le M(\varepsilon)$

$I \subset \{1, \dots, a\} \times \{1, \dots, b\}$ (bad pairs), $\sum_{(i,j) \in I} |V_i||W_j| \le \varepsilon |V||W|$,

s.t. $\forall (i,j) \notin I\ \exists d_{ij} \in [0,1]$ s.t. $\forall A \subseteq V_i, B \subseteq W_j$

$\big| |E \cap (A \times B)| - d_{ij}|A||B| \big| \le \varepsilon |V_i||W_j|$

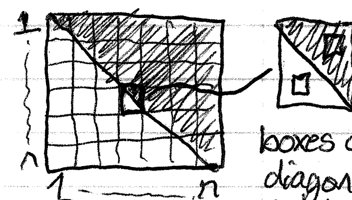
Examples:



Impressive part: This is very general, works for all E, all A and B. With more assumptions, one can make more precise statements.

Why bad pairs?

Half-graph (= order property)



Algebraic regularity lemma

F finite field, V, W definable over F, with complexity $\le C$

$d \le M(C) \quad (V \subseteq F^d,\ V = \{x \in F^d \mid \varphi(x) \text{ true}\})$

$E \subset V \times W$ definable, complexity $\le C$

$\Rightarrow V = V_1 \cup \dots \cup V_a \quad a \le M(C) \quad |V_i| \ge \frac{1}{M(C)}|V|$

$W = W_1 \cup \dots \cup W_b \quad b \le M(C) \quad |W_j| \ge \frac{1}{M(C)}|W|$

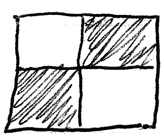
$\forall i,j,\ V_i, W_j$ definable, complexity $\le M(C)$

s.t. $\forall i \in \{1, \dots, a\}, j \in \{1, \dots, b\}\ \exists d_{ij} \in [0,1]$ s.t.

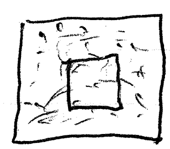
$\forall A \subset V_i\ \forall B \subset W_j,\ \big| |E \cap (A \times B)| - d_{ij}|A||B| \big| \le C|F|^{-1/4}|V_i||W_j|$

Examples: $V = W = F$

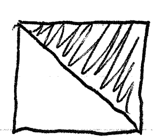
(Quadratic residues)

$E_1 = \{(v,w) \in F^2 \mid \exists t\ vw = t^2\}$ $\longrightarrow$ 

(Paley graph)

$E_2 = \{(v,w) \in F^2 \mid \exists t\ v - w = t^2\}$ $\longrightarrow$ 

Can't get the half-graph with bounded complexity



Proofs:

- Tao (char(F) suff. large) ← Chatzidakis, van den Dries, Macintyre; etale fundamental group $\pi_1(X)$
- Pillay-Starchenko + Hrushovski independently (all F) ← C, vD, M + stability theory
- Tao (all F) ← C, vD, M. + spectral theory

all proofs use Cauchy-Schwartz

Expanding polynomials

F finite field, $P: F \times F \to F$ bounded degree

When is P "expanding"? i.e. for $A \subseteq F$, $B \subseteq F$ } large, but not too large

$|P(A,B)| \gg |A|, |B|$

$P(A,B) = \{P(a,b) \mid a \in A, b \in B\}$

Nonexamples:

$P(x,y) = x+y$

$P(x,y) = x \cdot y$ (geometric series don't grow much)

(*) $P(x,y) = h(f(x)+g(y))$

(**) $P(x,y) = h(f(x) \cdot g(y))$

Thm P of bounded degree not of form (*) or (**).
Then if $|A|, |B| \geq |F|^{(1-1/16)} \Rightarrow |P(A,B)| \geq c|F|$

Look at $\{(x,y,P(x,y)) \mid x,y \in F\} \cap (A \times B \times C) \subseteq F^3$

↓ Cauchy-Schwarz

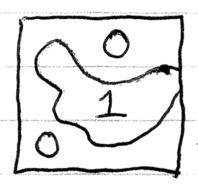
$\{(P(a,b), P(a,b'), P(a',b), P(a',b')) \mid a,b,c,d\} \cap (A \times B \times C \times D) \subseteq F^4$

Usually, this map $F^4 \to F^4$ is dominant...

but not always: $(a+b)-(a+b')-(a'+b)+(a'+b')=0$

Actually, this is a special case of Hrushovski's group configuration theorem! The only definable groups come from things like (*), (**)...

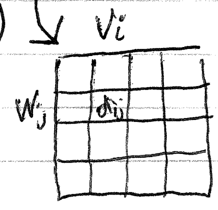
1_E:



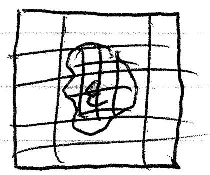
Regularity lemmas are compactness assertions.

$T: L^2(V) \to L^2(W)$

T "=" bounded rank + small error (in operator norm)



How to prove the regularity lemma?



$\varepsilon > 0$

Original strategy: Greedy algorithm.
Look for sets of density fluctuation, partition.
Hope the partitions have the regularity you want.
If not, density fluctuation, repeat.

Can't prove the algebraic case this way: Can't get as good bounds, and definitely can't get definability – density fluctuation sets could be arbitrary

Another proof: Spectral methods

$T : L^2(V) \to L^2(W)$, singular value decomp.

$$T = \sum_{i=1}^{N} \sigma_i V_i^* W_i, \quad \sum \sigma_i^2 \le 1$$

Another proof: TT^* method (Cauchy-Schwarz)

Express TT^* = bounded rank + small error

We need to use something about finite fields: $\boxed{CvDM}$

V definable over F, complexity $\le C$

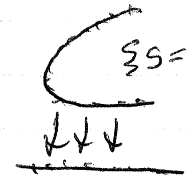
– $|V| = \alpha |F|^d + O_C(|F|^{d-\frac{1}{2}})$, $d \in \mathbb{N}$, $d \le C$, $\alpha > 0$, $\alpha \in \mathbb{Q}$ of height $\le M(C)$

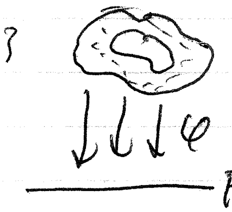
Lang-Weil inequality

– $V = \varphi(W(F))$ $\quad (V_w)_{w \in W}$

alg. variety of complexity $\le M(C)$

regular map of complexity $\le C$





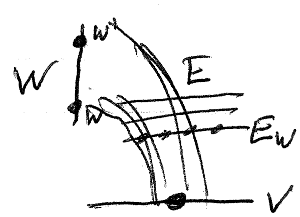
$V = W(F)$, $\alpha \in \mathbb{N}$, $d = \dim W$, α = # irr. top dim. components of W that are Frobenius-invariant

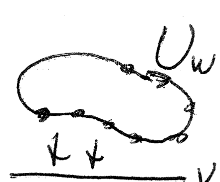
Idea:

Let $T = (1_E)$

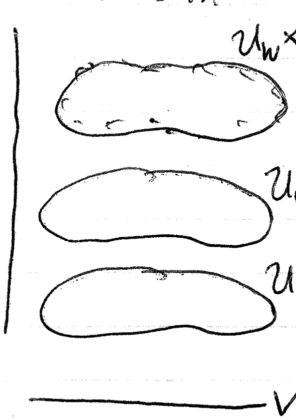
$\downarrow$

TT^* $\quad \mu(w, w')$





$$\mu(w,w') = |\{v \in V \mid (v,w), (v,w') \in E\}| = |E_w \cap E_{w'}|$$



$$|U_w \times_V U_{w'}(F)| = C_{w,w'} |F|^d + O(|F|^{d-\frac{1}{2}})$$

bounded rank

$$\begin{tikzcd} & U_w \times_V U_{w'} \arrow[dl] \arrow[dr] & \\ U_w \arrow[dr] & & U_{w'} \arrow[dl] \\ & V & \end{tikzcd}$$

Information:

$\pi_1(V) \circlearrowright$ fibre of U_w
Fibre of $U_{w'}$ $\longleftarrow$ $\pi_1(V)$ is topologically finitely generated (SGA4)
Frob, $\circlearrowright$ fibres

Pillay-Starchenko: You don't need all this algebraic geometry.

$|E_w \cap E_{w'}|$ Probability space (X, μ)

Lemma of Hrushovski
$(E_i)_{i\in\mathbb{N}}$ $(F_i)_{i\in\mathbb{N}}$
$\mu(E_i \cap F_j) > p$ gives a stable relation

If $(E_i)_{i=1}^N, (F_j)_{j=1}^N$ with $0 < q < p < 1$

$$\left.\begin{array}{ll}\mu(E_i \cap F_j) > p & i > j \\ \mu(E_i \cap F_j) < q & i < j\end{array}\right\} \Rightarrow N \le C(p,q)$$

Hrushovski: Proof via indiscernibles; exchangeability, De Finetti
Tao: Direct proof using Hilbert's inequality

$$\left\|\begin{pmatrix} 0 & & \\ & \frac{1}{i-j} & \\ & & 0\end{pmatrix}\right\|_{op} \le \pi, \quad \left|\sum_{i=1}^N \sum_{j=1}^N \frac{1}{i-j} 1_{E_i}(x) 1_{F_j}(x)\right| \le N\pi$$

$$N \log N (q-p) \le \left|\sum_{i=1}^N \sum_{j=1}^N \frac{1}{i-j} \mu(E_i \cap F_j)\right| \le N\pi$$