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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Martin Hils

Talk Title: A Model Theoretic Approach to Berkovich Spaces (III)

Date: 02/07/14 **Time:** 9:30 am / pm (circle one)

List 6-12 key words for the talk: Berkovich spaces, Hrushovski-Loeser spaces, strong deformation retraction, skeleta, internality

Please summarize the lecture in 5 or fewer sentences: Part 3 of 3. Boardwork with slides at the end of the talk. Definably compact (pro-definable sets: A variety is complete iff its Hrushovski-Loeser space is definably compact. Strong deformation retraction of Hrushovski-Loeser curves onto piecewise linear "skeleta" internal to the value group. Returning to Berkovich spaces from the Hrushovski-Loeser perspective.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Recall: If X is a pro-definable space (e.g. $X=\hat{V}$), $q\in S_{def,X}$, then $a\in\mathcal{U}$ is a limit of q ($\lim(q)=a$) if $q\vdash x\in\Omega$ for every open definable nbhd Ω of a (here $a\in X(\mathcal{U})$).

We've seen: $Z \overset{\text{prodef}}{\subseteq} \hat{V}\times\Gamma_\infty^n$ (e.g. $Z=\hat{X}$) is closed iff it is closed under def. limits.

Def: Z is definably compact if every $q\in S_{def,Z}$ has a limit in Z.

Prop: If $f: Z\longrightarrow \hat{W}\times\Gamma_\infty^n$ is prodef. and continuous, then $f(Z)$ is def. compact, provided Z is def. compact.

Pf: Key Lemma: X, Y any definable or pro-definable sets in ACVF. If $f: X\twoheadrightarrow Y$ is a surjective pro-definable map, then $f_{def}: S_{X,def}\twoheadrightarrow S_{Y,def}$, $p\mapsto f_*p$ is surjective, and $\hat{f}: \hat{X}\twoheadrightarrow\hat{Y}$ is surjective.

From the key lemma, it's an easy exercise.

Question: Did you define $\hat{X}$ for X pro-definable?
Answer: No, but one can do it. I have to put some things under the rug.

Thm: $Z\overset{\text{prodef}}{\subseteq}\hat{V}\times\Gamma_\infty^n$ is def. compact iff Z is closed and bounded.

Def: Let $V\overset{cl}{\subseteq}\mathbb{A}^n$ a variety/K.
- $X\overset{def}{\subseteq}V$ is bounded in V if $X\subseteq c\cdot\mathcal{O}^n$, where $c\in K$.
- If $V=\bigcup_i\mathcal{U}_i$, $(\mathcal{U}_i)$ an affine cover, then $X\overset{def}{\subseteq}V$ is bounded in V if $\exists X_i\subseteq\mathcal{U}_i$ bounded in $\mathcal{U}_i$ s.t. $\bigcup_i X_i = X$
- $Z\subseteq\hat{V}$ is bounded in $\hat{V}$ if $\exists X\overset{def}{\subseteq}V$ bounded in V s.t. $Z\subseteq\hat{X}$.
- $X\subseteq\Gamma_\infty^n$ is bounded if $\exists\gamma\in\Gamma$ s.t. $X\subseteq[\gamma,\infty]^n$.

Examples: (1) $\mathbb{P}^n$ is bdd in itself, so every $Z\subseteq\mathbb{P}^n$ is bdd.
($\mathbb{P}^n(K)=\bigcup_{i=0}^n\mathcal{U}_i(\mathcal{O}_K)$)
(2) $\mathbb{A}^n$ is unbounded in itself, but bounded in $\mathbb{P}^n$.

<u>Cor</u>: If V is an alg. variety, then $\hat{V}$ is def. compact iff V is complete.
<u>Pf</u>: Exercise. Uses Chow's Lemma and the theorem above.

<u>Retraction onto Γ-internal skeleta (for curves)</u>
From now on, V is a quasi-projective variety.

<u>Def</u>: $Z \subseteq \hat{V} \times \Gamma_\infty^{\wedge}$ is <u>Γ-internal</u> if there is $X \subseteq^{def} \Gamma_\infty^N$ and $f: X \underset{def}{\cong} Z$.

<u>Ex</u>: $\Gamma_\infty^n \hookrightarrow \widehat{\mathbb{A}^n}$, $(\gamma_1, \dots, \gamma_n) \longmapsto (p_{B_{\geq \gamma_1}(0)} \otimes \dots \otimes p_{B_{\geq \gamma_n}(0)})$ is def. and a homeomorphism onto its image (which is Γ-internal).

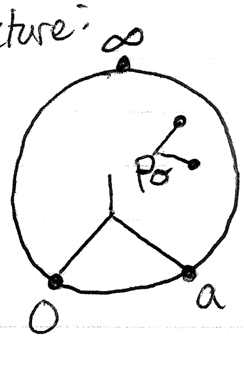
<u>Fact</u>: Let $f: C' \to C$ be a finite alg. map between curves C' and C, and let $\Sigma \subseteq \hat{C}$ be Γ-internal. Then $\hat{f}^{-1}(\Sigma)$ is Γ-internal.

<u>Prop</u>: (topological characterization of Γ-internal subsets)
Let $\Sigma \subseteq \hat{V} \times \Gamma_\infty^{\wedge}$ be Γ-internal. Then there is a continuous injection $\Sigma \hookrightarrow \Gamma_\infty^N$. If Σ is def. compact, f is automatically a homeomorphism onto its image.

Question: Why do we need definable compactness.
Answer: No counterexample known, but we don't know how to prove the stronger statement.

<u>Def</u>: A <u>generalized interval</u> $I = [o_I, e_I]$ in Γ_∞ is obtained as a concatenation of finitely many closed intervals of $(\Gamma_\infty, <)$ or $(\Gamma_\infty, >)$.

Picture:



Any two points of $\widehat{\mathbb{P}^1}$ are endpoints of a unique segment (generalized interval).

Def: • Let $I = [o_I, e_I]$ be a gen. interval. Then any continuous $f: I \times \widehat{X} \to \widehat{Y}$ is called a def. homotopy between $H_0: \widehat{X} \to \widehat{Y}$ and $H_e: \widehat{X} \to \widehat{Y}$.

• $H: I \times \widehat{X} \to \widehat{X}$ is a strong deformation retraction (onto $\Sigma \subseteq \widehat{X}$) if H is a def. homotopy s.t.

(1) $H_0 = id_{\widehat{X}}$

(2) $H\restriction_{I \times \Sigma} = id\restriction_{I \times \Sigma}$

(3) $im(H_e) \subseteq \Sigma$

(4) $H_e(H(t,a)) = H_e(a)$ $\forall t \in I$ $\forall a \in \widehat{X}$.

Thm Let C be an alg. curve. Then there is a strong def. retraction of $\widehat{C}$ onto $\Sigma \subseteq \widehat{C}$ Γ-internal and homeomorphic to some $X \subseteq \Gamma_\infty^N$.

Example: $\widehat{\mathbb{P}^1}$

There is a def. ultrametric $\mathbb{P}^1(K) \times \mathbb{P}^1(K) \xrightarrow{d} \Gamma_\infty$ bounded by 0 s.t.

$\widehat{\mathbb{P}^1} \longleftrightarrow \{$generic types of closed balls $p_{B_{\geq \gamma}(a)}$ of radius $\gamma \geq 0\}$

$H^{st}: [\overset{e_I}{0}, \overset{o_I}{\infty}] \times \widehat{\mathbb{P}^1} \longrightarrow \widehat{\mathbb{P}^1}$, $(t, p_{B_{\geq\gamma}(a)}) \longmapsto p_{B_{\geq \min(\gamma, t)}(a)}$

Fact: H^{st} is a strong def. retraction onto $\{p_0\}$.

Idea (general curve C): We may assume C is projective.

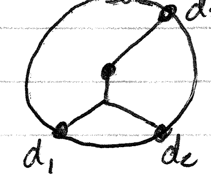
We may assume $\exists f: C \to \mathbb{P}^1$ finite and generically étale.

Definition of deformation of $\widehat{\mathbb{P}^1}$ with stopping time given a divisor $D \subseteq \mathbb{P}^1(K)$

C_D = convex hull of $D \cup \{p_0\}$ in $\widehat{\mathbb{P}^1}$

C_D is a finite closed Γ_∞-tree.

(In particular, it is Γ-internal)



Define: $H^D: [0,\infty] \times \widehat{\mathbb{P}^1} \to \widehat{\mathbb{P}^1}$

which does the same thing as H^{st}, until a point reaches C_D, when it stops.

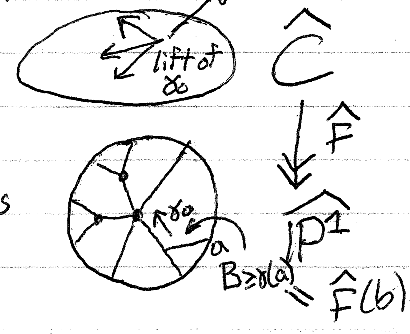
Check: H^D is a strong def. retraction onto C_D.

Show: * Any path like γ_0 lifts to some γ_n starting at b

* Problem: There may be more than one (gen of) such lifts

Say $\widehat{f}(b)$ is a forward branching point if this happens

* Key finiteness result: There are only finitely many f.b. pts.

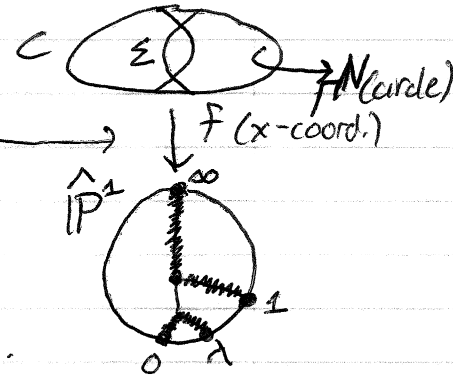


Choose $D \subseteq^{fin} \mathbb{P}^1(K)$ s.t.

- f is étale above $\mathbb{P}^1 \setminus D$
- C_D contains all f.b. pts.

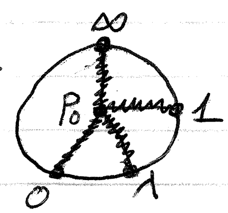
Not too difficult: H^D lifts uniquely to a strong def. retraction of $\widehat{C}$ onto $\underbrace{\widehat{f}^{-1}(C_D)}_{\Sigma\ (\Gamma\text{-internal})}$.

<u>Ex</u>: $y^2 = x(x-1)(x-\lambda)$, $0 < val(\lambda) < \infty$,
Ramified over $\{0, 1, \lambda, \infty\}$

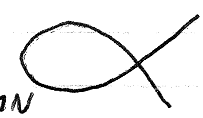


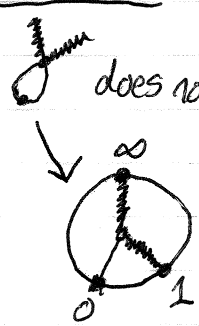
One shows: A point in $\widehat{\mathbb{P}}^1$ has two preimages iff it is not on the thick segment (where $|\widehat{f}^{-1}(p)| = 1$). Retracts to a circle.

<u>1st variant</u>: E_λ: $val(\lambda) = 0 = val(\lambda - 1)$.
$\times \subseteq \widehat{E_\lambda}$, $\widehat{E_\lambda}$ is contractible



<u>2nd variant</u>: $E_{\lambda=0}$, $y^2 = x^2(x-1)$





does not homeomorphically embed into Γ^N

Rest of the talk on slides.

The main theorem of Hrushovski-Loeser

Theorem

Suppose $A \subseteq \mathcal{U} \cup \Gamma(\mathcal{U})$. Let V be a quasiprojective variety and $X \subseteq V \times \Gamma_\infty^n$ an A-definable subset.

Then there is an A-definable strong deformation retraction $H : \mathrm{I} \times \widehat{X} \to \widehat{X}$ onto a Γ-internal subset $\Sigma \subseteq \widehat{X}$ such that Σ A-embeds homeomorphically into Γ_∞^w for some finite A-definable w.

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Corollary

Let X be as above. Then $\widehat{X}$ has finitely many definable connected components, all semi-algebraic and definably path-connected.

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Proof.

Let H and Σ be as in the theorem. By o-minimality, Σ has finitely many def. connected components $\Sigma_1, \ldots, \Sigma_m$. The properties of H imply that $H_e^{-1}(\Sigma_i) = \widehat{X_i}$, where $X_i = H_e^{-1}(\Sigma_i) \cap X$ □

Some words about the proof of the main theorem

- Proof by induction on $d = \dim(V)$, **fibering into curves**.

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- One shows the existence of H respecting extra data, namely
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- Only elementary tools from algebraic geometry are used, apart from Riemann-Roch (used the proof of iso-definability of $\widehat{C}$).
- Technically, the most involved arguments are needed to guarantee the continuity of certain homotopies. There are nice specialisation criteria (both for the v- and for the g-topology) which may be formulated in terms of 'doubly valued fields'.

Berkovich spaces revisited

- Let F a complete valued field such that $\Gamma_F \leq \mathbb{R}$.
- Set $\mathbb{F} = (F, \mathbb{R})$, where $\mathbb{R} \subseteq \Gamma$.
- Let V be a variety defined over F.
- Let X be an $\mathbb{F}$-definable subet.
- Denote by X^{an} the corresponding semialgebraic subset of V^{an}.

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Fact

As sets, we have the following canonical identification:

$$\{p \in S_X(\mathbb{F}) \mid p \text{ is almost orthogonal to } \Gamma\} = X^{an}.$$

Passing from $\widehat{X}$ to X^{an}

Given $\mathbb{F} = (F, \mathbb{R})$ as before, let $F^{max} \models \mathrm{ACVF}$ be maximally complete such that

- $\mathbb{F} \subseteq (F^{max}, \mathbb{R})$;
- $\Gamma_{F^{max}} = \mathbb{R}$, and
- $\mathbf{k}_{F^{max}} = \mathbf{k}_F^{alg}$.

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Lemma

The restriction of types map

$$\pi : \widehat{X}(F^{max}) \to S_X(\mathbb{F}), \; p \mapsto p|\mathbb{F}$$

induces a surjection $\pi : \widehat{X}(F^{max}) \twoheadrightarrow X^{an}$.

The topological link to actual Berkovich spaces

Proposition

1. *The map $\pi : \widehat{X}(F^{max}) \twoheadrightarrow X^{an}$ is continuous and closed. In particular, if $F = F^{max}$, it is a homeomorphism.*
2. *Any continuous prodefinable map $g : \widehat{X} \to \widehat{Y}$ defined over $\mathbb{F}$ descends to a (unique) continuous map*

$$\tilde{g} : X^{an} \to Y^{an}.$$

3. *Similarly, any prodefinable strong deformation retraction $H : I \times \widehat{X} \to \widehat{X}$ defined over $\mathbb{F}$ descends to a (unique) strong deformation retraction*

$$\tilde{H} : I(\mathbb{R}_{\infty}) \times X^{an} \to X^{an}.$$

The main theorem phrased for Berkovich spaces

Theorem

Let V be a quasiprojective variety defined over F, and let X be an $\mathbb{F}$-definable subset of V.

Then there is a strong deformation retraction

$$H : \mathrm{I}(\mathbb{R}_\infty) \times X^{an} \to X^{an}$$

onto a subspace **Z** *which is homeomorphic to a finite simplicial complex.*

Topological tameness properties for Berkovich spaces

Theorem

Let V be quasi-projective and definable over F.

1. V^{an} *is* **locally contractible**.
2. *Let X be an $\mathbb{F}$-definable subset of $V \times \mathbb{P}^n$ Then there are only* **finitely many homotopy types** *for X_b^{an}, where $b \in V$.*
3. *If V^{an} is compact, then it is homeomorphic to $\varprojlim_{i \in I} \mathbf{Z_i}$, where the $\mathbf{Z_i}$ form a projective system of subspaces of V^{an} which are homeomorphic to finite simplicial complexes.*
4. *Let $d = \dim(V)$, and assume that F contains a countable dense subset for the valuation topology. Then V^{an} embeds homeomorphically into $\mathbb{R}^{2d+1}$ (Hrushovski-Loeser-Poonen).*

- E. Hrushovski, F. Loeser. *Non-archimedean tame topology and stably dominated types.* arXiv:1009.0252.
- A. Ducros. *Les espaces de Berkovich sont modérés, d'après E. Hrushovksi et F. Loeser.* Séminaire Bourbaki, exposé **1056**, June 2012.
- M. Hils, *Tameness in non-archimedean geometry through model theory*. Slides for a tutorial at the Ravello meeting *Model theory 2013*, available at http://www.logique.jussieu.fr/∼hils/.