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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** kruckman@gmail.com

Speaker's Name: Ya'acov Peterzil

Talk Title: O-Minimal Ingredients in Proofs of Arithmetical Conjectures such as Manin-Mumford and Andre-Oort (II)

Date: 02/07/14 **Time:** 1:30 am / pm (circle one)

List 6-12 key words for the talk: o-minimality, Pila-Wilkie, special points, Ax-Lindemann, Manin-Mumford, André-Oort

Please summarize the lecture in 5 or fewer sentences: Part 2 of 2. Slides (22-42, pp. 121-224 in attached pdf) with supporting boardwork. Further applications of the Pila-Zannier method outlined in the previous talk: the Manin-Mumford conjecture, and the André-Oort conjecture. Ax-Lindemann in these contexts, a "naive" definition of Shimura variety, and the status of general André-Oort.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Reminder of the setting

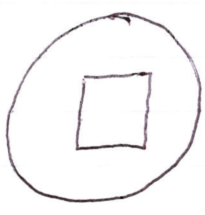
$$G \curvearrowright \tilde{V} \subseteq \mathbb{C}^n \xrightarrow{\ \Theta\ } V, \ \Gamma\text{-invariant}$$

$\Gamma \leq G$ infinite discrete

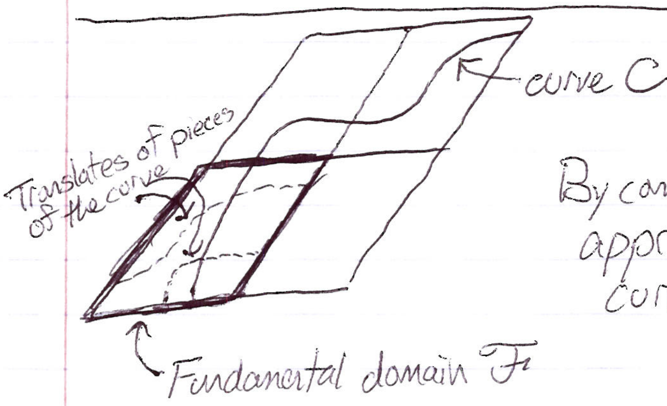
$\tilde{X} \subseteq \tilde{V}$ is $\widetilde{\text{special}}$ if

① $\tilde{X} = H \cdot z$, $H < G$ real algebraic
② $\Theta(\tilde{X})$ algebraic
③ $\tilde{X} \cap \tilde{S}_0 \neq \emptyset$ ← special points

} just ① + ②: $\widetilde{\text{weakly special}}$

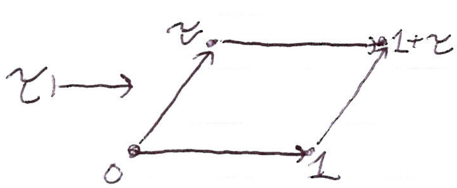


$f|_{\bigcirc}$ is analytic.
Then $f|_{\square}$ is definable (in fact in the basic language!) in $\mathbb{R}_{an}$, which is o-minimal.



By compactness / o-minimality, the translates approach a line as we go to ∞ along the curve.

$\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ as a parameter space of elliptic curves:

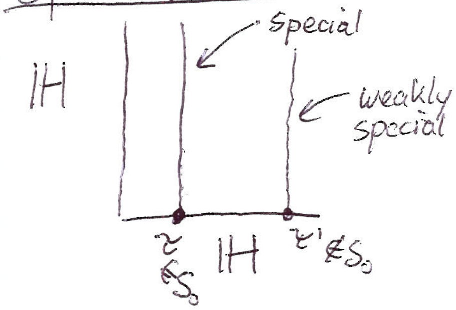


Lattice L_τ, $E_\tau = \mathbb{C}/L_\tau$

Recall, we start with $G(\mathbb{R})/K$. In the case of André-Oort for $\mathbb{C}^n$, this is $G(\mathbb{R})/stab_G(z)$.

$\mathbb{H}^n \xrightarrow{\ \theta = J^n\ } \mathbb{C}$, $\quad \Gamma = SL(n, \mathbb{Z})^n$

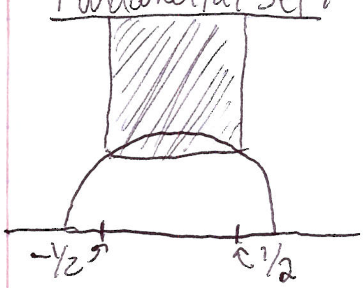
Special varieties:



and curves (can't get weakly special here – can't vary them in families)

Trivial (disintegrated) structure from the model theory point of view.

Fundamental set:



J invariant under $z \longmapsto z+1$

e^z is definable on a horizontal strip (last time)

$\Rightarrow e^{2\pi i z}$ is definable on this vertical strip

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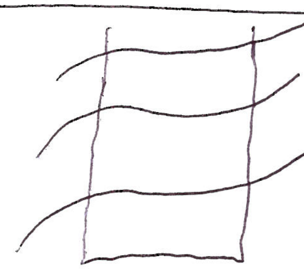
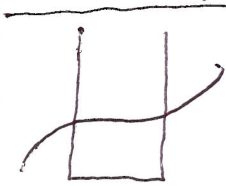
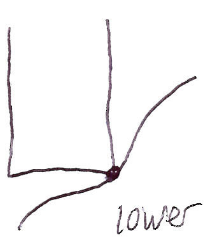
$\theta^{-1}(X)$ may have infinitely many connected components, but only finitely many go through the fundamental set (by definability)





full dimension intersection



lower dimension intersection

There are infinitely many γ giving rise to a full dimension intersection.

O-minimality and Arithmetic.
The Pila-Zannier method

Kobi Peterzil

Department of Mathematics
University of Haifa

MSRI workshop, February 2014

Some Bibliography

J. Pila and U. Zannier, *Rational points in periodic analytic sets and the Manin-Mumford conjecture,* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 19 (2008), no. 2, 149-162.

J. Pila, O-minimality and the André–Oort conjecture for $\mathbb{C}^n$. Ann. of Math. (2), 172(3), 2011, 1779–1840.

Survey papers

T. Scanlon, *A proof of the André–Oort conjecture via mathematical logic [after Pila, Wilkie and Zannier]*, Sèminaire BOURBAKI Avril 2011 63ème année, 2010–2011, no 1037.

T. Scanlon, *Counting special points: Logic, diophantine geometry, and transcendence theory*, Bull. AMS (N.S.) 49 (2012), no. 1, 51 – 71.

A general problem scheme

Setting

$\mathcal{C}$ = an underlying family of sets

$\mathcal{S} \subseteq \mathcal{C}$ is a collection of so-called "special" $\mathcal{C}$-sets

$\mathcal{S}_0$ = a set of so-called "special" points, often these are the $\mathcal{S}$-sets of dimension zero.

The problem scheme

Start with an $\mathcal{S}$-set V and consider an arbitrary $\mathcal{C}$-set $X \subseteq V$. Assume that X has "many" special points. Then X contains a special set of positive dimension. Under additional assumptions, X itself is a special set.

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The Pila-Wilkie results (viewed in this scheme)

Fix $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, ... \rangle$ an o-minimal expansion of the real field.

$\mathcal{C} =$ the family of all definable sets in $\mathcal{M}$.

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The Pila-Wilkie theorem(s)

Assume that $X \subseteq \mathbb{R}^n$ is definable in $\mathcal{M}$. If $X \cap (\mathbb{Q}^{alg})^n$ is *large** then X contains a connected infinite semi-algebraic set defined over $\mathbb{Q}$. More precisely, if one removes **all** infinite connected semi-algebraic subsets of X then a *small** number of $\mathbb{Q}^{alg}$-points remains.

$X \cap (\mathbb{Q}^{alg})^n$ is **large*** if exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that

$$limsup_T \frac{|\{\bar{q} \in X \cap (\mathbb{Q}_k^{alg})^n : height_k(\bar{q}) \leq T\}|}{T^\epsilon} = \infty.$$

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$\mathcal{S}_0$ = 0-dimensional $\mathcal{S}$-sets: special points.

V = an irreducible $\mathcal{S}$-variety.

$X \subseteq V$ = an irreducible complex algebraic subvariety

Assumption

The set $X \cap \mathcal{S}_0$ is Zariski dense in X.

Goal

The variety X is itself in $\mathcal{S}$.

From now on-the algebraic general problem scheme

The algebraic presentation

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A test case-the exponential example (algebraic torus)

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Let $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$ (so here V admits the structure of an algebraic group, which is also a complex Lie group).

$\mathcal{C} = \{X \subseteq (\mathbb{G}_m)^n : X$ an irreducible algebraic variety$\}$.

$\mathcal{S} = \{A + p : A$ a conn. algebraic subgrp of $\mathbb{G}_m^n$ & p a torsion point$\}$.

$\mathcal{S}_0 =$ Torsion points in $(\mathbb{G}_m)^n$

Theorem (Laurent)

If $X \subseteq (\mathbb{G}_m)^n$ an irreducible algebraic variety and $X \cap Tor(\mathbb{G}_m)^n$ is Zariski dense in X then $X = A + p$ for some $A \leqslant (\mathbb{G}_m)^n$ and $p \in Tor(\mathbb{G}_m)^n$.

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Let $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$ (so here V admits the structure of an algebraic group, which is also a complex Lie group).

$\mathcal{C} = \{X \subseteq (\mathbb{G}_m)^n : X$ an irreducible algebraic variety$\}$.

$\mathcal{S} = \{A + p : A$ a conn. algebraic subgrp of $\mathbb{G}_m^n$ & p a torsion point$\}$.

$\mathcal{S}_0 =$ Torsion points in $(\mathbb{G}_m)^n$

Theorem (Laurent)

If $X \subseteq (\mathbb{G}_m)^n$ an irreducible algebraic variety and $X \cap Tor(\mathbb{G}_m)^n$ is Zariski dense in X then $X = A + p$ for some $A \leqslant (\mathbb{G}_m)^n$ and $p \in Tor(\mathbb{G}_m)^n$.
Namely,
If $X \in \mathcal{C}$ and $X \cap \mathcal{S}_0$ is Zariski dense in X then $X \in \mathcal{S}$.

Back to the general problem-the analytic presentation

We work with affine (or projective) algebraic variety V and an algebraic subvariety $X \subseteq V$.

An analytic covering map

We have $\widetilde{V}=$ a (semi-algebraic) open subset of $\mathbb{C}^n$ (with $n = \dim V$). And $\Theta : \widetilde{V} \to V$ a holomorphic, **transcendental,** surjection.

General strategy

Replace V and its algebraic variety $X \subseteq V$ by $\widetilde{V}$ and a complex analytic subvariety $\Theta^{-1}(X) \subseteq \widetilde{V}$.

Caution

In general, Θ and $\Theta^{-1}(X)$ are not definable in any “tame” structure. We will need to “truncate” them.

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The analytic presentation: additional data

An underlying group action

We have $G =$ a real algebraic group acting semi-algebraically and transitively on $\widetilde{V}$. In some cases $\widetilde{V} = G$.

Γ= an infinite discrete subgroup of G (not necessarily normal).

The map $\Theta : \widetilde{V} \to V$ is Γ-invariant. Namely, $\Theta(x) = \Theta(y)$ if and only if $\Gamma x = \Gamma y$.

So, V can be identified with $\Gamma \backslash \widetilde{V}$.

If $X \subseteq V$ is a complex algebraic subvariety then $\Theta^{-1}(X) = \widetilde{X}$ is a Γ-invariant analytic subvariety of $\widetilde{V}$.

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$\widetilde{special}$-varieties and points

An irreducible analytic subvariety $Y \subseteq \widetilde{V}$ **is called a $\widetilde{\text{special}}$ variety** if $\Theta(Y)$ is a spacial subvariety of V. In particular, $\Theta(Y)$ is algebraic (**!**).

A point $z \in \widetilde{V}$ is **$\widetilde{\text{special}}$** if $\Theta(z)$ is a special point. Namely $\Theta(z) \in \mathcal{S}_0$.

Fact (an alternative definition): $\widetilde{\text{special}}$ varieties as orbits

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Recall $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$. Take $\widetilde{V} = \mathbb{C}^n$ and $\Theta := \exp : \mathbb{C}^n \to (\mathbb{C}^*)^n$ defined by $exp(z_1, \ldots, z_n) = (e^{z_1}, \ldots, e^{z_n})$.

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Because Θ is a homomorphism, $\Theta(z)$ is a torsion point of order k iff $kz \in \Gamma$. So, $\widetilde{\mathcal{S}_0} = \{\bar{z} \in \mathbb{C}^n : \exists k \; k\bar{z} \in (2\pi i\mathbb{Z})^n\} = (2\pi i\mathbb{Q})^n$.

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The general case: The (non) definability of Θ

We have $\Theta : \widetilde{V} \to V \quad \sim \Gamma\backslash\widetilde{V}$

The non-definability of Θ

Since Γ is infinite discrete, $\Theta^{-1}(p)$ is an infinite discrete set (for every $p \in V$). Hence, the map Θ is never definable in an o-minimal structure.

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The ingredients for the Pila-Zannier method

We have $\Theta : \widetilde{V} \to V \sim \Gamma \backslash \widetilde{V}$. $\mathcal{S}_0 \subseteq V$ the set of special points.

I. Definability requirements (from algebraic to o-minimal)

One needs to establish the existence of a semialgebraic fundamental set $\mathfrak{F} \subseteq \widetilde{V}$ and the definability of $\Theta \restriction \mathfrak{F}$ in some o-minimal structure $\mathcal{M}$. In all examples, $\mathcal{M}$ is $\mathbb{R}_{an,exp}$.

For $X \subseteq V$ algebraic, let $\widetilde{X} \subseteq \widetilde{V}$ be an irreducible analytic component of $\Theta^{-1}(X)$. Note that $\widetilde{X} \cap \mathfrak{F} = (\Theta \restriction \mathfrak{F})^{-1}(X)$ is definable in $\mathcal{M}$.

II. Number theory goal

- The set $\widetilde{\mathcal{S}}_0 = \Theta^{-1}(\mathcal{S}_0)$ is contained in $\mathbb{Q}_k^{alg}$ for some k(up to definable bijection).

▶▶ If $X \cap \mathcal{S}_0$ (on the algebraic side) is Zariski dense in X then $\widetilde{\mathcal{S}}_0 \cap (\widetilde{X} \cap \mathfrak{F})$ (on the analytic side) is large* (in the sense of Pila-Wilkie). This is "the lower bound".

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- Assume that we established that $\widetilde{\mathcal{S}}_0 \cap (\widetilde{X} \cap \mathfrak{F})$ is large*.
- By PW, There exists a connected semi-algebraic nontrivial curve $C \subseteq \widetilde{X} \cap \mathfrak{F}$.
- Let $\overline{C} \subseteq \mathbb{C}^n$ be the Zariski closure of C. It is a complex algebraic curve, and by dimension considerations $(\overline{C} \cap \widetilde{V}) \subseteq \widetilde{X}$.
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- Let $\overline{C} \subseteq \mathbb{C}^n$ be the Zariski closure of C. It is a complex algebraic curve, and by dimension considerations $(\overline{C} \cap \widetilde{V}) \subseteq \widetilde{X}$.
- So $\widetilde{X}$ **contains a complex algebraic curve** (relative to the open semialgebraic $\widetilde{V}$).

The ingredients of the Pila-Zannier method (cont)

The Pila-Wilkie input

- Assume that we established that $\widetilde{\mathcal{S}}_0 \cap (\widetilde{X} \cap \mathfrak{F})$ is large*.
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The Pila-Zannier method: The punch-line!

We have $\Theta : \widetilde{V} \to V \sim \Gamma\backslash\widetilde{V}$. $\widetilde{X} \subseteq \widetilde{V}$ a component of $\Theta^{-1}(X)$.

The general idea

Take $\widetilde{A}$ a maximal algebraic subset of $\Theta^{-1}(X)$.
The Γ-periodicity of $\Theta^{-1}(X)$ together with the algebraicity of $\widetilde{A}$ is "unlikely" and should imply that the stabilizer of $\widetilde{A}$ in $G(\mathbb{R})$ is nontrivial.
In fact, it should imply that $\widetilde{A}$ is "special".

More precisely,

Ingredient III, the "Ax-Lindemann" goal

Assume that $\widetilde{A}$ is a maximal irreducible algebraic (relative to $\widetilde{V}$) subset of $\widetilde{X}$.
Then $\widetilde{A}$ is a weakly $\widetilde{\text{special}}$ variety. Namely,

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Summary of the Pila-Zannier method

We have $X \subseteq V$, $\Theta : \widetilde{V} \to V$ and $X \cap \mathcal{S}_0$ Zariski dense in X.

I. Definability

$\Theta \restriction \mathfrak{F}$ is definable in an o-minimal structure.

II. Number Theory

The set $\widetilde{S_0} \cap (\Theta^{-1}(X) \cap \mathfrak{F})$ is large*.

Application of the Pila-Wilkie Theorem.

III. Ax-Lindemann

If $\widetilde{A} \subseteq \Theta^{-1}(X)$ is maximal irreducible algebraic then it is weakly special. (So, f in addition $\widetilde{A} \cap \widetilde{\mathcal{S}_0} \neq \varnothing$ then $\widetilde{A}$ is special).

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Applying Pila-Zannier to the exponential case

We have $V = (\mathbb{C}^*)^n$

$\mathcal{S}_0 = Tor(\mathbb{C}^*)^n$

$\mathcal{S} = \{A + p : A \leqslant (\mathbb{C}^*)^n \ , \ p \in \mathcal{S}_0\}.$

I. Fundamental set and the definability of Θ

We have $\Theta : \mathbb{C}^n \to (\mathbb{C}^*)^n$ given by $\Theta(z_1, \ldots, z_n) = (e^{z_1}, \ldots, e^{z_n})$.

- A fundamental set for Θ is:

$$\mathfrak{F} = \{\bar{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : 0 \leqslant |Im(z_i)| \leqslant \pi\}.$$

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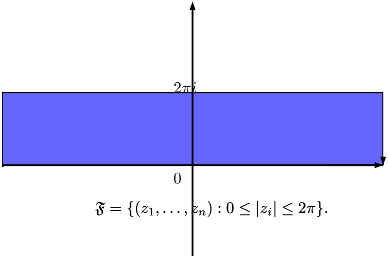
The exponential case: the definability of $\Theta \restriction \mathfrak{F}$



$\mathfrak{F} = \{(z_1, \ldots, z_n) : 0 \leq |z_i| \leq 2\pi\}.$

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We have $e^z = e^{x+iy} = e^x(cosy + i\, siny)$.

The map e^x is definable in $\mathbb{R}_{exp}$; the maps $cos, sin \restriction [0, 2\pi]$ are definable in $\mathbb{R}_{an}$, hence:

The map $e^z \restriction \{0 \leqslant Im(z) \leqslant 2\pi\}$ is definable in $\mathbb{R}_{an,exp}$.

It follows that $\Theta \restriction \mathfrak{F} = exp \restriction \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

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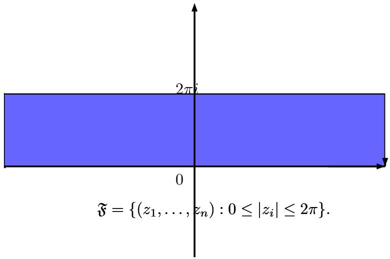

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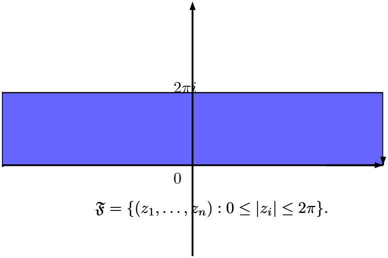

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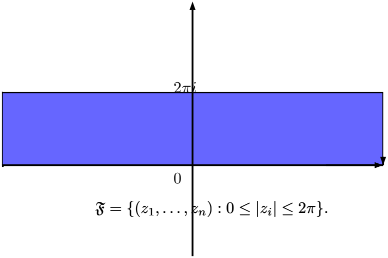

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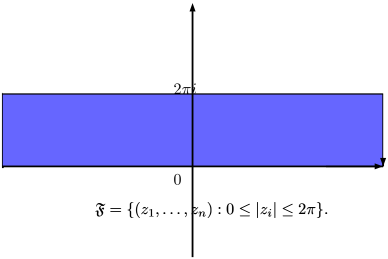

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The exponential case: II. Basic Number Theory

- Assume that $X \subseteq (\mathbb{C}^*)^n$ is an irreducible algebraic variety and that $X \cap Tor(\mathbb{C}^*)^n$ is Zariski dense in X.

We want to show that $(2\pi i\mathbb{Q})^n \cap (\Theta^{-1}(X) \cap \mathfrak{F})$ is large*.

- X is defined over a number field k. For simplicity, $k = \mathbb{Q}$.
- Since $X \cap Tor(\mathbb{C}^*)^n$ is infinite there are natural numbers $m_1 < m_2 < \dots$ and elements $g_i \in X$, with $ord(g_i) = m_i$.
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Fact For every $0 < \epsilon < 1$, $\lim \phi(m)/m^\epsilon = \infty$.

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The following set is large*

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The exponential case: II. Basic Number Theory

• Assume that $X \subseteq (\mathbb{C}^*)^n$ is an irreducible algebraic variety and that $X \cap Tor(\mathbb{C}^*)^n$ is Zariski dense in X.

We want to show that $(2\pi i\mathbb{Q})^n \cap (\Theta^{-1}(X) \cap \mathfrak{F})$ is large*.

• X is defined over a number field k. For simplicity, $k = \mathbb{Q}$.

• Since $X \cap Tor(\mathbb{C}^*)^n$ is infinite there are natural numbers $m_1 < m_2 < \ldots$ and elements $g_i \in X$, with $ord(g_i) = m_i$.

• If $g \in (\mathbb{C}^*)^n$, and $ord(g) = m$ then g has at least $\phi(m)$ conjugates over $\mathbb{Q}$, where $\phi(m) = \#\{i \leqslant m : (i, m) = 1\}$ is the Euler function.

Fact For every $0 < \epsilon < 1$, $\lim \phi(m)/m^\epsilon = \infty$.

Hence, $\lim_i \frac{|\{g \in X : ord(g) = m_i\}|}{m_i^{1/2}} = \infty$.

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The exponential case: the Ax Lindemann statement

The Pila-Wilkie input

The analytic set $\Theta^{-1}(X) \subseteq \mathbb{C}^n$ contains a nontrivial algebraic set. Take a maximal such irreducible algebraic set A.

Goal: **A is weakly special** $=$ a coset of a linear s.space of $\mathbb{C}^n$ over $\mathbb{Q}$.

A proof using the classical Ax's theorem (corrected)

Ax's Theorem If $\xi_1, \ldots, \xi_n \in \mathbb{C}(A)$ and $lin.dim_{\mathbb{Q}}(\bar{\xi}/\mathbb{C}) = m$ then $tr.deg(\mathbb{C}(exp(\xi_1), \ldots, exp(\xi_n))/\mathbb{C}) = m$.

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It follows that $\Theta(A)$ is Zariski dense in $\Theta(H) + \Theta(p)$, so $\Theta(H) + \Theta(p) \subseteq X$.

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• We started with $X \subseteq (\mathbb{G}^m)^n$ such that $Tor(\mathbb{G}_m)^n \cap X$ is Zariski dense in X.

• Using Pila-Wilkie, we concluded that $\Theta^{-1}(X)$ contained a nontrivial complex algebraic set A. Furthermore we can choose it so $A \cap \widetilde{\mathbb{S}}_0$ is nonempty. Take such A maximal.

• By Ax, A is weakly $\widetilde{\text{special}}$, hence $\widetilde{\text{special}}$ ($A \cap \widetilde{\mathbb{S}}_0 \neq \varnothing$).

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Another example: The Manin-Mumford conjecture

The setting

V = an abelian variety in $\mathbb{P}^n(\mathbb{C})$.

So, V is a projective algebraic variety which admits an algebraic group structure, abelian. It is also a compact, complex Lie group.

$\mathcal{C}$ = all irreducible algebraic subvarieties of V.

$\mathcal{S}$ = all cosets of the form $A + p$, where $p \in Tor(V)$ and A a connected algebraic subgroup (i.e. abelian subvariety) of V.

$\mathcal{S}_0 = Tor(V)$ the torsion elements.

The Manin-Mumford conjecture (Raynaud's Theorem, 1983)

Assume that V is a complex abelian variety defined over a number field, and $X \subseteq V$ an irreducible algebraic subvariety. If $X \cap Tor(V)$ is Zariski dense in V then $X = A + p$ as above.

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The analytic presentation

- There exists a holomorphic group homomorphism $\Theta : (\mathbb{C}^n, +) \to V$.
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- **Weakly special varieties** are arbitrary cosets of such H.

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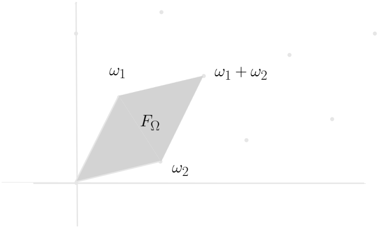
The Pila-Zannier method for Manin-Mumford

I. The fundamental set and definability of $\Theta \restriction \mathfrak{F}$

Consider the compact semilinear parallelogram
$\mathfrak{F} = \{\sum_{i=1}^{2n} t_i\omega_i : 0 \leqslant t_i \leqslant 1\}$. Then:
(i) $\Gamma + \mathfrak{F} = \mathbb{C}^n$.
(ii) The set $\{\gamma \in \Gamma : (\gamma + \mathfrak{F}) \cap \mathfrak{F} \neq \varnothing\}$ is finite.
$\mathfrak{F}$ is a fundamental set for Θ.



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I. The fundamental set and definability of $\Theta \restriction \mathfrak{F}$

Consider the compact semilinear parallelogram
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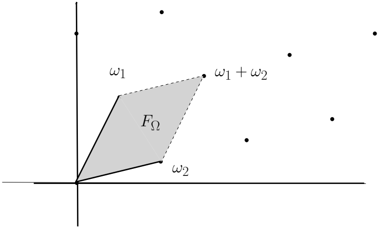

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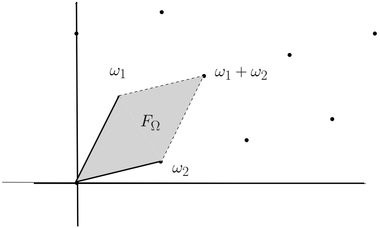

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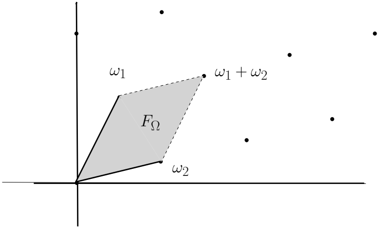

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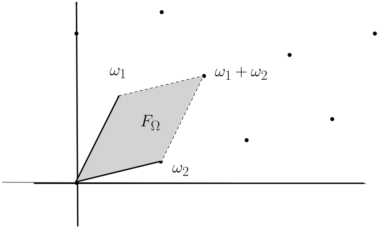

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II. Number Theory (on the algebraic side)

- V is an abelian variety defined over a number field F.
- $X \subseteq V$ is irreducible algebraic, with $X \cap Tor(V)$ Zariski dense in X.
- So, X is also defined over a number field $k \supseteq F$.

Number theoretic input (Masser)

There exists $\rho = \rho(V) > 0$ and a constant c, such that for every $P \in V$, if $ord(P) = T$ then $[F(P) : \mathbb{Q}] \geqslant cT^{\rho}$.

By conjugating $X \cap Tor(V)$ over k we conclude: if $\epsilon < \rho(V)$ then

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The set $\{(q_1, \ldots, q_{2n}) \in \mathbb{Q}^{2n} : \sum_{j=1}^{2n} q_j \omega_i \in \Theta^{-1}(X) \cap \mathfrak{F}\}$ is **large***.

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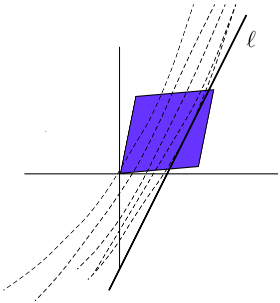
III. Ax-Lindemann: an o-minimal argument

The Pila-Wilkie input

The analytic variety $\Theta^{-1}(X)$ contains an unbounded semialgebraic curve σ.

By the o-minimality of σ, when we translate it into $\mathfrak{F}$ by elements of Γ we get (inside $\widetilde{X}$) curves which are more and more "linear". Since $\widetilde{X} \cap \mathfrak{F}$ is compact, at the limit we get an affine line $\ell \subseteq \widetilde{X}$.



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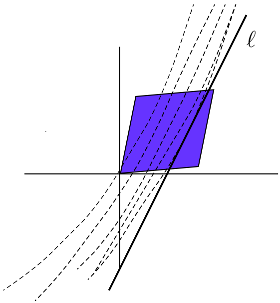

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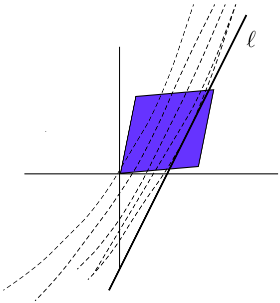

Finishing the proof of MM

On the analytic side

We saw that $\Theta^{-1}(X)$ contains an affine line $\ell \subseteq \mathbb{C}^n$.

Back to the algebraic side

The variety $X \subseteq V$ contains $\Theta(\ell)$, a coset of a subgroup.

The Zariski closure of $\Theta(\ell)$ is a coset of an algebraic subgroup of V, which is contained in X.

Hence, X contains a (weakly) special variety $z + A$, for $A \leqslant X$.

By using the full strength of Pila-Wilkie, together with the ability to write V as a an almost direct product $A \oplus B$, we can show that X itself is a special variety.
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The general analytic setting for Shimura varieties (simplified)

- $G(\mathbb{R})$ is the $\mathbb{R}$-points of an algebraic semisimple group G over $\mathbb{R}$.
- $K \leqslant G(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$.
- (with additional assumptions) the quotient space $G(\mathbb{R})/K$ admits the structure of an open semi-algebraic subset of $\mathbb{C}^n$. This set is our $\widetilde{V}$.
- $G(\mathbb{R})$ acts on $\widetilde{V}$. Actually, for every $g \in G(\mathbb{R})$, $g : \widetilde{V} \to \widetilde{V}$ is a biholomorphism.
- Let $\Gamma = G(\mathbb{Z})$ (more generally, an arithmetic subgroup), and consider the quotient $V = \Gamma \backslash \widetilde{V}$.

The Baily-Borel Theorem (1966)

There exists a holomorphic embedding $\Theta : \Gamma \backslash \widetilde{V} \to \mathbb{P}^m(\mathbb{C})$ whose image is a quasi-projective variety.

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There exists a holomorphic embedding $\Theta : \Gamma \backslash \widetilde{V} \to \mathbb{P}^m(\mathbb{C})$ whose image is a quasi-projective variety.

$Im(\Theta) = V$ is ***a Shimura variety*** (a non-specialist viewpoint).

Andre-Oort setting

The general analytic setting for Shimura varieties (simplified)

- $G(\mathbb{R})$ is the $\mathbb{R}$-points of an algebraic semisimple group G over $\mathbb{R}$.
- $K \leqslant G(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$.
- (with additional assumptions) the quotient space $G(\mathbb{R})/K$ admits the structure of an open semi-algebraic subset of $\mathbb{C}^n$. This set is our $\widetilde{V}$.
- $G(\mathbb{R})$ acts on $\widetilde{V}$. Actually, for every $g \in G(\mathbb{R})$, $g : \widetilde{V} \to \widetilde{V}$ is a biholomorphism.
- Let $\Gamma = G(\mathbb{Z})$ (more generally, an arithmetic subgroup), and consider the quotient $V = \Gamma \backslash \widetilde{V}$.

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The Shimura variety $\mathbb{C}^n$: Preliminaries

We start with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$.

The group $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ (transitively) as follows:
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\tau \in \mathbb{H}$ then $A \cdot \tau = \frac{a\tau+b}{c\tau+d}$.

Connection to elliptic curves

$\mathbb{H}$ is a parameter space for elliptic curves, namely, every τ represents the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ where Λ_τ the lattice $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$.

$E_{\tau_1} \cong E_{\tau_2} \Leftrightarrow \tau_1, \tau_2$ are in the same $SL(2, \mathbb{Z})$-orbit. So, $SL(2, \mathbb{Z})\backslash\mathbb{H}$ is the moduli space of elliptic curves.

The J-invariant

There exists a holomorphic, transcendental surjection $J : \mathbb{H} \to \mathbb{C}$ such that $J(\tau_1) = J(\tau_2) \Leftrightarrow SL(2, \mathbb{Z})\tau_1 = SL(2, \mathbb{Z})\tau_2$.

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Special varieties and points

Again, the definition begins on the analytic side.

Definition of special points: The set $\widetilde{\mathbb{S}}_0$

$(\tau_1, \ldots, \tau_n) \in \mathbb{H}^n$ is **special**, if for every i, the elliptic curve E_{τ_i} has complex multiplication ($End(E_\tau) \neq \mathbb{Z}$).
Equivalently, τ_i belongs to an imaginary quadratic extension of $\mathbb{Q}$.
(abstract definition of special points in Shimura varieties-omitted here).

Definition of special varieties

Recall: An irreducible analytic variety $Y \subseteq \mathbb{H}^n$ is **special** if
(i) Y is an orbit of a real algebraic group $H \leqslant SL(2, \mathbb{R})^n$.
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- $\widetilde{X} = \{\tau\} \times \mathbb{H}^{n-1}$, with $\tau \in \widetilde{\mathcal{S}}_0$; it is an orbit of $H = \{1\} \times SL(2,\mathbb{R})^{n-1}$. $\Theta(\widetilde{X}) = \{p\} \times \mathbb{C}^{n-1}$ is a special variety.
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The statement of the theorem

The André-Oort Conjecture for $\mathbb{C}^n$ (a theorem of Pila)

If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety and $X \cap \mathcal{S}_0$ is Zariski dense in X then X is special.

By the nature of the definitions, we immediately have an analytic presentation of the problem:

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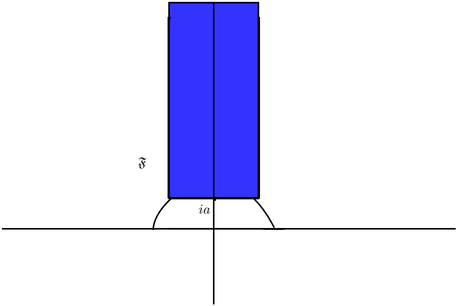
The Pila Zannier method: I. The fundamental set

By the basic theory of elliptic curves, the following is a fundamental set for $SL(2,\mathbb{Z})$ (for every $0 < a < \sqrt{3}/2$):

$$\mathfrak{F} = \{z \in \mathbb{H} : -1/2 \leqslant Re(z) \leqslant 1/2 \,\&\, Im(z) > a\}.$$



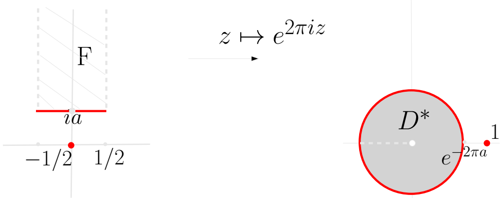
So $\mathfrak{F}^n$ is a fundamental set for $SL(2,\mathbb{Z})^n$.

Pila-zanner method I: Definability of $J \restriction \mathfrak{F}$

Theorem

The restriction of J to $\mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

Proof Consider first the map $z \mapsto e^{2\pi iz}$. It sends $\mathfrak{F}$ onto a punctured disc D^*. The "point" $Im(z) = \infty$ is sent to 0.



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- As pointed out in an earlier talk, we may write J in the variable $q = e^{2\pi iz}$ and obtain a meromorphic function on D^*. Hence (???) $J(q)$ is definable in $\mathbb{R}_{an}$. It follows that $J(z)$ is definable in $\mathbb{R}_{an,exp}$.

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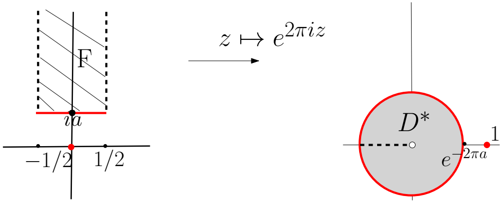

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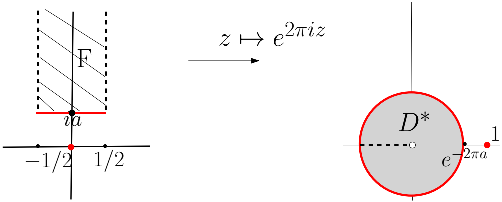

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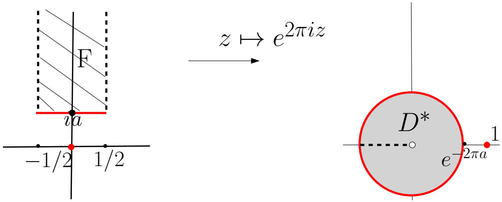

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We have $\Theta : \mathbb{H}^n \to \mathbb{C}^n$, and $X \subseteq \mathbb{C}^n$ algebraic, with $X \cap \mathcal{S}_0$ Zariski dense in X. We use $\mathfrak{F}$ for the fundamental set for Θ ($= \mathfrak{F}^n$).

On the analytic side

Let $\widetilde{X} \subseteq \mathbb{H}^n$ be an irreducible **analytic** component of $\Theta^{-1}(X)$.
We already saw that if $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{H}^n$ is special then each τ_i is imaginary quadratic, so $\widetilde{\mathcal{S}}_0 \subseteq (\mathbb{Q}_2^{alg})^n$..

Using a theorem of Siegel on imaginary quadratic fields, Pila proves:

Largness of special points

The set $\widetilde{\mathcal{S}}_0 \cap \widetilde{X} \cap \mathfrak{F}$ is large*.

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III. The Ax-Lindemann statement

The Pila-Wilkie input

$\widetilde{X}$ contains an algebraic set of positive dimension (relative to $\mathbb{H}^n$). Let A be maximal irreducible such set.

Goal

A is weakly special. Namely
(i) it is the orbit of a real algebraic subgroup of $SL(2, \mathbb{R})^n$, and
(ii) $\Theta(A)$ is algebraic.

Ax-Lindemann for $\mathbb{H}^n$ (third type of proof)

• We have $A \subseteq \Theta^{-1}(X)$ a maximal, irreducible relatively algebraic subset, of positive dimension. Namely, there exists an algebraic $\bar{A} \subseteq \mathbb{C}^n$ such that $A = \bar{A} \cap \mathbb{H}^n$.

Write $G := SL(2,\mathbb{R})^n$, and $\Gamma = SL(2,\mathbb{Z})^n$.

• Without loss of generality $\dim(A \cap \mathfrak{F}) = \dim A$ (if not, replace $\widetilde{X}$ and A by $\gamma\widetilde{X}$ and γA, for some $\gamma \in \Gamma$).

Fact A is not contained in finitely many Γ-translates of $\mathfrak{F}$.

WHY?

Otherwise $A \subseteq \bigcup_{i=1}^{k} \gamma_i \mathfrak{F}$. Because the real part of $\mathfrak{F}$ is bounded, it follows that $Re(z)$ is bounded for $z \in \bar{A} \cap \mathbb{H}^n$. This would imply (?) that A must be compact. But a compact complex analytic subset of $\mathbb{H}^n$ is finite. Contradiction.

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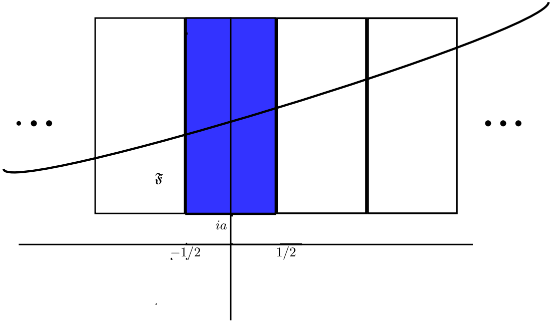

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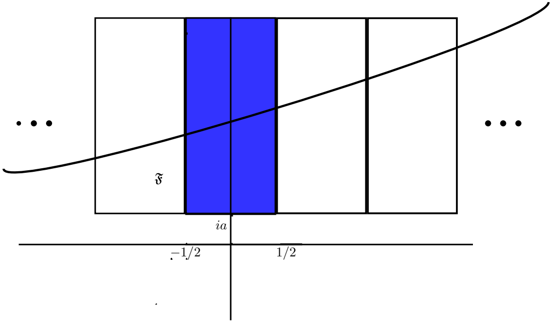

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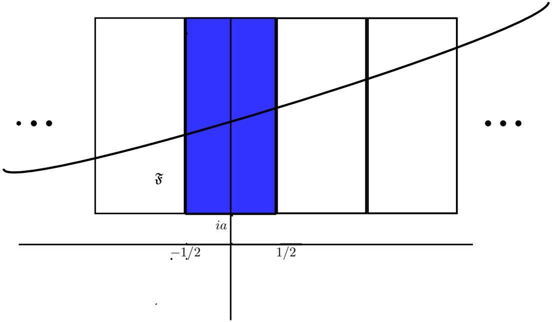

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- The set $G(A)$ is definable in $\mathbb{R}_{an,exp}$.

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The set $\{\gamma \in SL(2,\mathbb{Z})^n : \gamma \in G(A)\}$ is large*.

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By PW, $G(A)$ contains a semi-algebraic connected curve σ.

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• Consider the real algebraic group $Stab_G(A) \subseteq G$. It is thus infinite and contains infinitely many Γ points (by a finer use of Pila-Wilkie).

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André Oort for $\mathcal{A}_g$ for $g = 2$ (Pila Tsimerman)

Theorem The André- Oort conjecture holds for $\mathcal{A}_2$, the moduli space of abelian surfaces.

I. Definability: P-Starchenko.

II. Number Theory: Uses results of Tsimerman.

III. A-L: using strongly the low dimension of $\mathcal{A}_2$ ($\dim \mathcal{A}_2 = 3$).

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Recent work of Klingler, Yafaev and Ullmo (2013)

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André Oort for $\mathcal{A}_g$ for $g = 2$ (Pila Tsimerman)

Theorem The André- Oort conjecture holds for $\mathcal{A}_2$, the moduli space of abelian surfaces.

I. Definability: P-Starchenko.

II. Number Theory: Uses results of Tsimerman.

III. A-L: using strongly the low dimension of $\mathcal{A}_2$ ($\dim \mathcal{A}_2 = 3$).

Status of General André-Oort

Recent work of Klingler, Yafaev and Ullmo (2013)

I. The restriction of the Baily-Borel embedding of any Shimura variety to the Siegel fundamental set is definable in $\mathbb{R}_{an,exp}$ (!).
III. Ax-Lindemann holds for arbitrary Shimura varieties.

what is missing?

The number Theory part