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Mathematical Sciences Research Institute

17 Gauss Way   Berkeley, CA 94720-5070   p: 510.642.0143   f: 510.642.8609   www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Alice Medvedev

Talk Title: Algebraic Dynamics and the Model Theory of Difference Fields

Date: 02/07/14 **Time:** 2:30 am / pm (circle one)

List 6-12 key words for the talk: dynamical Mordell-Lang, dynamical Manin-Mumford, special points, difference fields, orthogonality, Zilber trichotomy

Please summarize the lecture in 5 or fewer sentences: Boardwork with 2 slides. An introduction to questions in algebraic dynamics from the perspective of special points and special subvarieties: dynamical Manin-Mumford and dynamical Mordell-Lang. The model theory of difference fields and contributions of the ideas of geometric stability theory.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Algebraic Dynamics $F : X \to X$

X: naive variety or scheme, maybe reducible
F: regular or rational morphism, usually dominant/quasifinite

Orbit: $\mathcal{O}_F(a) := \{F^{\circ n}(a) : n \in \mathbb{N}\}$
where $F^{\circ 0}(a) := a, F^{\circ(n+1)}(a) = F(F^{\circ n}(a))$

Fixed point: $F(a) = a$; periodic: $F^{\circ n}(a) = a$;
preperiodic: $F^{\circ(m+n)}(a) = F^{\circ n}(a)$.

Subvariety $Y \subset X$ is invariant if $F(Y) \subset (Y)$;
usually $F : Y \to Y$ dominant required.
Periodic: $F^{\circ n}(Y) \subset Y$; and preperiodic: $F^{\circ(m+n)}(Y) = F^{\circ n}(Y)$.

"Geometric Staibility" in a difference field (L, σ)

$\sigma : L \to L$ is a field automorphism, $X^\sigma := \{\sigma(a) : a \in X\}$.
σ-variety: $F : X \to X^\sigma$, dominant.

Sub-σ-variety: $Y \subset X$ such that $F : Y \to Y^\sigma$ is dominant.

(X, F) is *orthogonal* to (Y, G) if every sub-σ-variety of $(X, F) \times (Y, G)$ is a union of components of products of sub-σ-varieties of (X, F) and (Y, G).

(X, F) is *disintegrated* if any irreducible component of sub-σ-variety Z of any cartesian power $(X, F)^{\times n}$ is a component of

$$\cap_{i,j \le n} \pi_{i,j}^{-1}(\pi_{i,j}(Z))$$

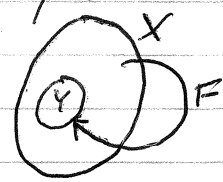
where $\pi_{i,j}$ are coordinate projections to $(X, F) \times (X, F)$.

(in the algebraic geometric category)

We're doing discrete dynamics. $F: X \to X$, what happens when we iterate F?

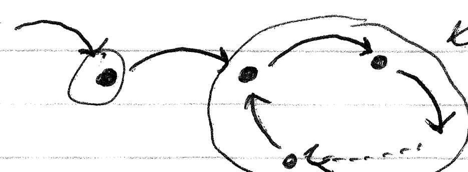
Taking F to be rational instead of regular really introduces new problems — if we remove the places where F isn't defined, this introduces new places F isn't defined (the preimage of this set)

Why can we take F to be dominant?



If Y is small, we won't ever leave a small set. So if F is not nearly surjective, we're looking at the wrong ambient variety.

Preperiodic point



Periodic points

$Y \subsetneq X$ invariant means $F(Y) \subseteq Y$, so we can consider $F|_Y : Y \to Y$.

- a is pre-periodic iff $\mathcal{O}_F(a)$ is finite
- Y is F-invariant $\Rightarrow$ top dim. irreducible components of Y are permuted by F, so are preperiodic.
- Some top dim. component(s) of Y must be periodic (if not all are periodic, F cannot be dominant $Y \to Y$).
- If Y is F-periodic, then Y is $F^{\circ n}$-invariant for some n (and conversely)

Ex: $X = \mathbb{G}_m$, $F(x) = x^2$. (No interesting subvarieties... but could take $(\mathbb{G}_m)^n$)

(pre) Periodic pts: roots of unity, i.e. torsion pts.

orbit of a: (infinite subset of) group generated by a

$X = \mathbb{G}_m^n$, $F(\bar{x}) = (x_1^2, \dots, x_n^2)$

① (pre) periodic pts: tuples of torsion pts.
② orbit of $\bar{a}$: (inf. subset of) grp. gen. by $\bar{a}$

} two options for "special pts" (roughly corresponding to MM and ML)

preperiodic subvarieties: torsion translates of subgroups

Specialness Principle:

$Y \subseteq X$ is F-special $\iff$ $\{F\text{-special pts}\} \cap Y$ Zariski-dense in Y

* This is a principle, not a theorem (except in some cases) *

F-special could mean preperiodic or invariant (for a variety)

1 (pre-periodic pts are special) dynamical Manin-Mumford

$\Rightarrow$: dominant morphism + ample magic — F. Fakhrudin

One definitely needs some conditions $\quad F^{\circ(m+n)}(a) = F^{\circ(n)}(a)$

Counterexample: $X = \mathbb{A}^1$, char 0, $F(x) = x+1$.

$\Leftarrow$: if in "special", we can bound m, n in the det. of preperiodic, this is cheap.

(Medvedev + Scanlon) - very special case

2 (orbits are special) dynamical Mordell-Lang - usually stated in a different way:

$\mathcal{O}_F(a) \cap Y \sim \{n \mid F^{\circ n}(a) \in Y\}$ is a finite union of arithmetic progressions

not necessarily special

What do finite unions of arithmetic progressions look like?

- finite set - nothing is said (arithmetic step size 0)
- $\mathcal{O}_{F^{\circ n}}(F^{\circ m}(a)) \subseteq Y$, if the arithmetic progression is $\{m + kn \mid k \in \mathbb{N}\}$

Note: $\overline{\mathcal{O}_F(a)}^{Zariski} = Z \subseteq X$ is F-invariant.

So there is $Z \subseteq Y$, $F^{\circ n}$-invariant, i.e. F-periodic. $(Z = \overline{\mathcal{O}_{F^{\circ n}}(F^{\circ m}(a))}^{Zar})$

$\Leftarrow$: Close to saying the orbit of a point intersect Y is controlled by some arithmetic progressions.

If $\mathcal{O}_F(a) \subseteq Y$, automatically get $Z \subseteq Y$ special, $\dim(Z) > 0$.

Many special case results: Bell, Benedetto, Ghioca, Hutz, Kurlberg, Tucker, Vinay, Zieve. $\boxed{\text{nice slides on dynamical ML}}$

$\Rightarrow$: Set a - unlikely to have $\mathcal{O}_F(a) \cap Y \neq \emptyset$

Instead, given a special subvariety $Y \subseteq X$, is there some a s.t. $\mathcal{O}_F(a) \cap Y$ is Zariski-dense in Y? (WLOG, this is about picking $a \in Y$. The ambient variety doesn't really matter. A bit contrived to put this in the special pts/special subvarieties setting.)

- i.e. find $a \in Y$ not on any special subvariety $Z \subsetneq Y$.

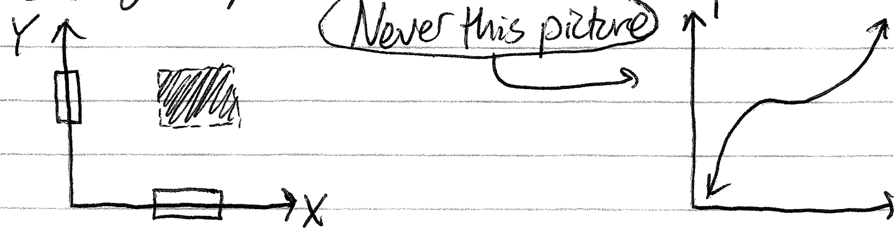
If Y is covered by such Z, this is hopeless

If a in some big field is Zariski dense/k in Y, this is free.

More interesting: find a in some small field.

- (Medvedev - Scanlon) - a special case.

Orthogonality:



Interesting issues: There might be reducible interactions between varieties where there are no irreducible interactions – one has to allow reducible varieties. An invariant reducible variety may not have irreducible components that are invariant – but they will be periodic. May have to go to higher powers of F. Should allow varieties to be defined over larger fields.

Zilber trichotomy. If there is no group or field you can get your hands on, things are very disintegrated, there aren't many families.