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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** kruckman@gmail.com

Speaker's Name: Lou van den Dries

Talk Title: Model Theory and Multiplicative Combinatorics (III)

Date: 02/07/14 **Time:** 4:00 am / pm (circle one)

List 6-12 key words for the talk: locally compact groups, logic topology, approximate groups, Breuillard-Green-Tao.

Please summarize the lecture in 5 or fewer sentences: Part 3 of 3. A presentation of the "logic topology" on the quotient of a countable union of definable sets by a type definable equivalence relation. In the case of definable groups, this gives rise to a locally compact topological group, which was used in the proof of the BGT theorem.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Extracting a locally compact group from a pseudofinite approximate group

1° The logic topology: extracts a locally compact space from logical data

2° Sanders-Croot-Sisask: result from multiplicative combinatorics (substitute for Hrushovski's stabilizer theorem)

3° Application of 1° & 2° to pseudofinite approximate groups

The Logic Topology

Idea: generalize the constuction of $\mathbb{R}$ as

$\mathbb{N}$-bounded elements of a big ordered field / infinitesimals

Notation: for a relation $R \subseteq P \times Q$ and $p \in P$, $R(p) := \{q \in Q : R(p,q) \text{ holds}\}$

$\mathcal{U}$: $\aleph_1$-saturated structure for a countable language

can be many-sorted

D: a definable set in $\mathcal{U}$

E: a Π-definable equivalence relation on D

means: $E = \bigcap_n E_n$, $E_n \subseteq D \times D$ definable

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Example: D a big ordered field,

$x E y :\Longleftrightarrow |x-y| \leq \frac{1}{n}$ for all $n \geq 1$

Using saturation we arrange

$$E_{n+1} \subseteq E_n, \quad E_{n+1}^{-1} \subseteq E_n, \quad E_{n+1} \circ E_{n+1} \subseteq E_n$$

$\therefore$ $\{E_n : n \in \mathbb{N}\}$ is a base of entourages for a <u>uniform structure</u> on D

We give D the topology induced by this uniform structure, so a point $a \in D$ has nbhd-base $\{E_n(a) : n \in \mathbb{N}\}$. The quotient space D/E is hausdorff

Assume also given a $\underbrace{\Sigma\text{-definable set}}_{\text{countable union of definable sets}}$ $S \subseteq D$

such that S is a union of E-classes. Then S is open in D. <u>Assume</u> : S is <u>E-bounded</u>, that is, for each n, S is covered by countably many sets $E_n(a)$, $a \in S$. For $S = \bigcup_m S_m$, where each S_m is definable, this means that each S_m is covered for each n by finitely many sets $E_n(a)$, $a \in S_m$. ("Each S_m is totally bounded")

In <u>Example</u>, these conditions hold for

$$S = \{\mathbb{N}\text{-bounded elements of } D\}$$

$S/E :=$ image of S in D/E
with the subspace topology

$\pi : S \longrightarrow S/E$. For $Y \subseteq S/E$:

Y open in $S/E \iff \pi^{-1}Y$ is Σ-definable

For this reason the topology of S/E is called the <u>logic topology</u> (Lascar, Pillay)

Unlike other topologies arising in logic, it has "classical" properties:

<u>Proposition</u> S/E is locally compact, 2^{nd} countable

For $Y \subseteq S/E$: Y compact $\iff \pi^{-1}Y$ is Π-definable

$X \subseteq S$ is Π-definable $\Rightarrow \pi X$ is compact

This applies to a definable group $\mathcal{G}$ in $\mathfrak{U}$ (in the role of D) as follows:

Let $\mathcal{G}^{\Sigma}$ be a Σ-definable subgroup of $\mathcal{G}$.

Let $\mathcal{G}^{\Pi}$ " " Π-definable " " " ,

such that $\mathcal{G}^{\Pi} \subseteq \mathcal{G}^{\Sigma}$:

$$\begin{array}{c} \mathcal{G} \\ | \\ \mathcal{G}^{\Sigma} \\ | \\ \mathcal{G}^{\Pi} \end{array}$$

Let E be the equivalence relation on $\mathcal{G}$ given by

$$x E y :\Longleftrightarrow x \in y\mathcal{G}^{\Pi}$$

Then E is Π-definable, and:

<u>Lemma</u> $\mathcal{G}^{\Sigma}$ is E-bounded

$$\Longleftrightarrow$$

for all definable $X, Y \subseteq \mathcal{G}^{\Sigma}$ with $X \supseteq \mathcal{G}^{\Pi}$, finitely many left translates of Y cover X

$\therefore$ Suppose G^{Σ} is E-bounded and $G^{\Pi} \trianglelefteq G^{\Sigma}$.

Then $\mathcal{G} := G^{\Sigma}/G^{\Pi} = G^{\Sigma}/E$ with its logic topology is a locally compact topological group. Let $\pi : G^{\Sigma} \longrightarrow \mathcal{G}$ be the canonical map. Then for definable $X \subseteq G$:

$X \supseteq G^{\Pi} \implies X \supseteq \pi^{-1}\mathcal{U}$ for some nbhd $\mathcal{U}$ of the identity in $\mathcal{G}$

In Example with $G =$ additive group of big ordered field,

$G^{\Sigma} = \{ \mathbb{N}\text{-bounded elements} \}$

$G^{\Pi} = \{ \text{infinitesimals} \}$

we get $G^{\Sigma}/G^{\Pi} \simeq \mathbb{R}$

Sanders, Croot - Sisask

They proved a result in multiplicative combinatorics:

Th Let $X \subseteq G$ be finite symmetric, $|X^2| \leq K|X|$. Let $m \geq 2$ be given. Then there exists a symmetric $S \subseteq G$ such $|S| \geq c|X|$, $S^m \subseteq X^4$ with $0 < c = c(K)$.

BGT derive from it a "normal" variant.

For $x, y \in G$, set $x^y := y^{-1}xy$

For $X, Y \subseteq G$, set $X^Y := \{x^y : x \in X, y \in Y\}$

Cor Let $X \subseteq G$ be a finite K-approximate group, $Y \subseteq X$ symmetric, $|Y| \geq \delta |X|$, $\delta > 0$.

Then there is a symmetric $S \subseteq G$ such that $|S| \geq \varepsilon |X|$, $(S^{16})^X \subseteq Y^4$, $\varepsilon = \varepsilon(K, \delta) > 0$.

<u>Informally</u>: for a finite K-appr. X and a significant portion Y of X we can find a significant portion S of X such that

$$(S^{16})^X \subseteq Y^4$$

Let now $X \subseteq G$ be a <u>pseudofinite</u> K-<u>approximate group</u>. Formally this is a 2-sorted $\mathcal{L}$ with underlying sets G and $\mathbb{R}^*$, with G equipped with its group law and the distinguished subset X, $\mathbb{R}^*$ a definably complete ordered field with distinguished set $\mathbb{N}^* \subseteq \mathbb{R}^*$ behaving like $\mathbb{N}$ in $\mathbb{R}$ (definably well-ordered), together with a distinguished bijection

$$X \xrightarrow{\sim} \{1, \dots, N\}, \quad N \in \mathbb{N}^*.$$

The "normalized" Sanders-Croot-Sisask theorem, applied iteratively, gives a descending sequence

$$X^4 = X_0 \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots \supseteq X_n \supseteq \cdots$$

of definable symmetric sets $X_n \subseteq G$ such that

$X_{n+1}^2 \subseteq X_n$, $X_{n+1}^X \subseteq X_n$, and

X^4 is covered by finitely many left translates of X_n, for each n.

$\therefore$ $G^\Pi := \cap X_n$ is a Π-definable normal subgroup of the Σ-definable group $G^\Sigma := \langle X \rangle$, ~~with~~ and G^Σ is E-bounded for the coset equivalence relation E on G corresponding to G^Π.

$\therefore$ get locally compact group

$$\mathcal{G} = \langle X \rangle / G^{\pi}, \quad \pi : \langle X \rangle \longrightarrow \mathcal{G}$$

and $X^4 \supseteq \pi^{-1}\mathcal{U}$ for some neighborhood of the identity in $\mathcal{G}$.