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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman Email/Phone: Kruckman@gmail.com

Speaker's Name: Anand Pillay

Talk Title: Stability theory and Diophantine geometry (II)

Date: 02/07/14 Time: 11:00 am / pm (circle one)

List 6-12 key words for the talk: Mordell-Lang, Manin-Mumford, jet spaces, theorem of the Kernel.

Please summarize the lecture in 5 or fewer sentences: Part 2 of 2. Slides (pp. 61-124 of attached pdf) with supporting boardwork. A "new" approach to Mordell-Lang, via differential jet spaces, and a reduction of Mordell-Lang to Manin-Mumford (the function field cases). A relevant "pure model theory" consideration: quantifier elimination for type-definable sets.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Correction to Thm 0.1: Only true for <u>finite-dimensional</u> minimal types.

X $b \in X$

$\text{tr.deg}\{\partial_{ij}(b) \mid i,j\}/K_0 < \omega$

$/K_0$ ↖ (tr.deg.)

$tp(b/K_0)$ is f.dim, or (thin)

In DCF_0, this is finite Morley rank / U-rank
In SCF_p, this implies finite U-rank, but is not equivalent to it.

(very thin) means everything is eventually <u>separably</u> algebraic over previous stuff.
jet spaces approach only works in this case.

---

jet spaces = linear spaces associated to sheafs of diff. operators

a on V, $(m/m^2)^* = TV_a$, $(m/m^{n+1})^* = j_n(V)_a$

$Y \subset V$ with $a \in Y$.
You know Y if you know $j_n(Y)_a \subseteq j_n(V)_a$ for all n.

$(Y_b : b \in Z)$, $Y_b \subseteq V$ with $a \in Y_b$. (b are canonical parameters, everything is up to definable bijection, i.e. birational equivalence)
Then $Z \hookrightarrow Gr(j_n(V)_a)$

Kolchin tangent space: $j_n^\partial(X)_a$

Non 1-basedness of $X \Rightarrow a \in X \times X$ generic, $(Y_b : b \in Z)$, $a \in Y_b$.

$Z \hookrightarrow Gr(j_n^\partial(X \times X)_a)$ ⤳ connection with the constants.

Algebraic analogue of complex analytic statement:
A complex torus $(\mathbb{C}/\Lambda)$
analytic $X \subset A$ irreducible with $Stab(X)$ finite
$\Rightarrow X$ <u>algebraic</u> variety.

$(Y-b : b \in Y)$ yields descent to the constants (Step III) in the same way.

Motivation: Work in bimeromorphic geometry (Campana)

$\partial(y) = s(y)$, $a \in V$ $\quad s$ ———— $\mathcal{U}$

↑ a point is a section

———— $V(\mathcal{U})$

$\partial_n y = A_n y$ Solution set is a f.dim. vector space over the absolute constants C.

↖ Should be $\partial_i y = A_i y$, A_i $n \times n$ matrices, where n is the dimension of the ambient variety.

Question: Is a torus ordinary?
Answer: A semiabelian variety is ordinary if its abelian part is ordinary.
A torus has no abelian part. (so yes)

Question: Abramovich-Voloch used Gauss maps. Is the proof different?
Answer: $Y \hookrightarrow Gr(-)$
$b \mapsto j_n(Y-b)_a$ is a version of the Gauss map.

Types $\longleftrightarrow$ Varieties (working with types smooths things out)
Canonical bases $\longleftrightarrow$ Chow points of varieties

---

$K < \mathcal{U}$ (Use of compactness/saturation in char $p > 0$ case)

↑ defined here ↑ defined here

$C_n \xrightarrow{\text{compactness}} C$

cosets

Stability and diophantine geometry
MSRI introductory workshop

Anand Pillay

University of Notre Dame

February 8, 2014

Introduction

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- I will not discuss explicitly the stability-theoretic approach to Manin-Mumford over number fields, although this has also had an impact on current developments (e.g. algebraic dynamics).
- I will try to convey something of the richness of the mathematics, although the stability-theoretic background and tools are less accessible to the "outsider" (or even "insider") than o-minimality.

Statements I

- The origins are (i) the Mordell conjecture that a curve X of genus > 1 over a number field F has only finitely many F-rational points, and (ii) a conjecture of Manin that a curve of genus > 1 embedded in its Jacobian variety $J(X)$ meets only finitely many torsion points of $J(X)$.

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- A big common generalization of (i) and (ii) is Mordell-Lang: (char. 0.) If G is a semiabelian variety, $\Gamma < G$ is "finite-rank", i.e. contained in the division points of a finitely generated subgroup Γ_0, X a subvariety of G and $X \cap \Gamma$ is Zariski-dense in X, then X is a translate of an algebraic subgroup of G. When Γ is the just the torsion subgroup of G, this is sometimes called Manin-Mumford.

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- M-L was proved by Faltings, McQuillan... M-M is "easier", was first proved by Raynaud and has many other proofs, including by Hrushovski, and more recently Pila-Zannier (both with model theoretic input).

Statements II

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- It is worthwhile giving the "geometric" description of the latter when $K = \mathbb{C}(t)$, as this is often how things are described in the literature.
- X is the general fibre of a family of complex algebraic curves $\mathcal{X} \to S$ where S is $\mathbb{P}^1(\mathbb{C})$ minus finitely many points, X not defined over $\mathbb{C}$ means the family is nonconstant, and K-rational points of X are precisely rational sections $S \to \mathcal{X}$.

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- **Function field ML in characteristic p**, as formulated by Abramovich and Voloch, is just as above, but the "finite rank" assumption on Γ is replaced by: Γ is contained in the prime-to-p division points of a finitely generated subgroup Γ_0 of $G(K)$. (And defined over k is meant in a weaker sense.)

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- So we consider proofs of the above two statements, where the characteristic p case is the truly new theorem.

Differentially closed fields

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- Differential algebra already played a role in work of Manin and of Buium on characteristic 0 function field ML.

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- Again $\mathcal{C}$ is an algebraically closed field with "no induced structure" (be careful as it is just type-definable).

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- Note that 1-basedness and "internality to ..." make sense for any type-definable X, whether or not X is minimal.
- Zilber actually formulated the principle in the special case when X is "strongly minimal" (namely minimal and definable).

Dichotomy II.

So Step I of the proof (connected with earlier work of Hrushovski, Zilber, Sokolovic, and of independent interest) is:

Theorem 0.1

The Zilber principle is true for $DCF_{0,m}$ and $SCF_{p,m}$ (for finite-dimensional, also called thin, minimal types). Moreover in either case F can be taken to be the field of (absolute) constants $\mathcal{C}$.

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- The proof goes via "Zariski geometries" (or structures). A Zariski geometry, as defined in HZ, or in Zilber's book is a strongly minimal set X (in an ambient structure if one wishes) such that certain subsets of $X, X \times X, ..$ are designated to be closed, and these closed sets satisfy abstract conditions somewhat like the Zariski closed sets of Cartesian powers of a smooth algebraic curve.

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- The HZ theorem is that the Zilber principle holds for Zariski geometries (and this theorem is the raison d'etre for the notion of Zariski geometries).

Dichotomy III

- All proofs of the HZ theorem are complicated to say the least, involving an abstract notion of tangency of closed definable sets as well as Hrushovski's group and field configuration theorem. I don't want to say anything more about it now but will discuss possible direct treatments in the case at hand in my second talk.

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- To prove Theorem 0.1 for $SCF_{p,m}$ by such methods requires dealing with *type-definable* Zariski geometries. These do not appear in Zilber's book and are not explicit in HZ. Nevertheless arguments are given for adapting HZ to this case and proving that thin, minimal sets in $SCF_{p,m}$ are Zariski.

Dichotomy IV

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Dichotomy IV

- A good exposition of type-definable Zariski geometries and the relevant applications would be welcome, and possibly appears already in Bernard Elsner's thesis.
- In any case, Theorem 0.1 is at the core of Hrushovski's approach to ML (although in our expositions 20 years ago we tended to take this as an unexplained black box).
- The remainder of the proof involves:
  Step II.: embedding the ML data into a definable framework (DCF_0/SCF_p),
  Step III. using stable-group-theoretic arguments together with Step I (Theorem 0.1) to obtain descent of the (type)-definable data to the constants, and
  Step IV.: deducing descent of the original algebraic geometric data.

Step II, characteristic 0

- So we have the data k, K, G, X, Γ. G,X, Γ are defined over the algebraic closure of $K_0 = k(t_1, .., t_m)$ for some m. Put the canonical partial differential structure on K_0 and pass to the differential closure which we may assume to be K and has constants k. (In fact we may take $m = 1$.)

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- Buium defines a generalized "logarithmic derivative" homomorphism μ (definable in the differential field K) from $G = G(K)$ to some $(K, +)^n$, whose kernel $G^\sharp$ has finite Morley rank.

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- Buium defines a generalized "logarithmic derivative" homomorphism μ (definable in the differential field K) from $G = G(K)$ to some $(K, +)^n$, whose kernel $G^\sharp$ has finite Morley rank.
- It follows that Γ is contained in a definable subgroup H of G which has finite Morley rank. Why? Then $X^\sharp = X \cap H$ is definable, and Zariski dense in X.

Step II, characteristic 0

- So we have the data k, K, G, X, Γ. G,X, Γ are defined over the algebraic closure of $K_0 = k(t_1, .., t_m)$ for some m. Put the canonical partial differential structure on K_0 and pass to the differential closure which we may assume to be K and has constants k. (In fact we may take $m = 1$.)
- Buium defines a generalized "logarithmic derivative" homomorphism μ (definable in the differential field K) from $G = G(K)$ to some $(K, +)^n$, whose kernel $G^\sharp$ has finite Morley rank.
- It follows that Γ is contained in a definable subgroup H of G which has finite Morley rank. Why? Then $X^\sharp = X \cap H$ is definable, and Zariski dense in X.
- The original data G, X, Γ has been replaced by the definable data G, X, H. Moreover $X^\sharp$ has trivial stabilizer in H.

Step II, characteristic p

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- For each n, $p^n\Gamma$ has finite index in Γ, hence some coset C_n of $p^nG(\mathcal{U})$, defined over K, meets X in a Zariski-dense set. We may assume the C_n's are compatible, hence by saturation of $\mathcal{U}$, $C = \cap_n C_n$, a coset of $G^\sharp =_{def} p^\infty G(\mathcal{U}) =_{def} \cap_n p^n G(\mathcal{U})$, type-definable over K, meets X in a Zariski-dense set.

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- Γ has been replaced by the type-definable group $G^\sharp$, which can be shown to be finite-dimensional, so finite U-rank (but not necessarily Morley rank).

Stable group theory 1

- We start with **Step III** in the char. 0 case, namely we have the finite Morley rank (commutative) group H definable in $K \models DCF_0$ and $X^\sharp$ is a definable subset of H with trivial stabilizer.

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- Udi proves the (weak) socle theorem for arbitrary finite Morley rank groups H (nice but not too hard)

Theorem 0.2

Suppose H is sufficiently "rigid" in the sense of having no infinite definable families of definable subgroups. Suppose Y is a definable subset of H with finite stabilizer. Then, up to to translation $Y \subseteq s(H)$.

Stable group theory 2

- Now, in general we can write $s(H) = H_1 + H_2$ where H_1 is generated by 1-based strongly minimal sets, and H_2 is generated by non 1-based strongly minimal sets. It follows that H_1 is 1-based in its own right and early work (HP) yields:

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- By Theorem 0.2, after a translation $X^\sharp$ is contained in $s(H)$.
- The **structure of 1-based groups** above and the assumption that $Stab(X^\sharp)$ is trivial implies that after a further translation $X^\sharp \subseteq H_2$.

Stable group theory 3

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- Step IV is obtained by taking Zariski closures of the (type) definable data. This involves additional arguments, especially in the characteristic p case where we want to deduce descent to k from descent to $\mathcal{C}$. But more or less straightforward. End of outline.

Although not made explicit in the sketch above, a key ingredient is:

Theorem 0.3

Suppose A is an abelian variety over $\mathcal{U}$ with $\mathcal{C}$-trace 0 then $A^\sharp$ is 1-based, and moreover (strongly) minimal when A is simple.

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- The "bad news" is that the approach does not always work in positive characteristic case, and only recovers Mordell-Lang for so-called ordinary semiabelian varieties, already done by Abramovich-Voloch.
- So in the first part of this second talk I will give a few details.

Differential jet spaces 1

- Suppose first that V is an (affine) algebraic variety over an algebraically closed field K, and $a \in V(K)$. We have the higher tangent spaces of V at a, namely $j_n(V)_a$ is the dual space to m/m^{n+1} where m is the maximal ideal of V at a namely the set of functions in the coordinate ring $K[V]$ of V, which vanish at a.

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- Exactly the same thing holds for "finite-dimensional" differential algebraic varieties, in characteristic 0 at least, which we will discuss next.

Differential jet spaces 2

- If X is defined by $f(y, y') = 0$ say then the Kolchin tangent space to X at a point a is defined by the linear differential equation $(\partial f/\partial y)(a)(u) + (\partial f/\partial y')(a)(u') = 0$.

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- For a differential algebraic variety X and $a \in X$ the higher Kolchin tangent spaces $j_n^{\partial}(X)_a$ (differential jet spaces) at $a \in X$ are defined by linear differential equations, and "finite-dimensionality" of X corresponds to these spaces being finite-dimensional vector spaces over the constants.

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- Hence **Fact 1**: If $(Y_b : b\in Z)$ is a differential algebraic family of differential algebraic subvarieties of (finite-dimensional) X, all passing through $a\in X$, we have a differential rational (so definable) embedding of Z in $Gr(j_n^\partial(X)_a)$ for some n.

Differential jet spaces 3.

- Now suppose X is a strongly minimal (so finite-dimensional) differential algebraic variety. Non 1-basedness of of X means precisely that there is an infinite definable family $(Y_b : b \in Z)$ of differential algebraic subvarieties of $X \times X$ passing through a generic point $a \in X \times X$.

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- Now $j^n(X \times X)_a$ is internal to $\mathcal{C}$, hence by Fact 1, so is Z. This yields some definable relationship between X and $\mathcal{C}$ which is enough to prove Theorem 0.1 for DCF_0.

In fact we also obtain:

Theorem 0.4

(Strong socle theorem) Let G be a finite Morley rank group in DCF_0 and Y a differential algebraic subvariety with trivial stabilizer. Then Y is internal to $\mathcal{C}$.

Differential jet spaces 4.

- The proof is: $(Y_b = Y - b : b \in Y)$ is a canonical definable family of differential algebraic subvarieties of G passing through 0, hence Y definably embeds in some $Gr(j_n^{\partial}(G)_0)$ so is internal to $\mathcal{C}$.
- This yields directly Step III of the proof in the characteristic 0 case.

Positive characteristic case 1

- The situation described above depends essentially on being able to describe a finite-dimensional differential algebraic variety in characteristic 0 (up to a change of coordinates) as the solution set of a first order polynomial differential equation $\partial(y) = s(y)$ on an algebraic variety V (so s is a kind of Ehresmann connection on V).

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- So in positive characteristic ($SCF_{p,1}$ say), the approach directly works for so-called ∂-varieties X, where ∂ is the iterative Hasse derivation.
- Namely sets X of the form $\{x \in V(\mathcal{U}) : \partial_n(x) = s_i(x) : i = 1, 2, ...\}$ for V an algebraic variety and s_n suitable polynomial functions.

Positive characteristic case 2

- So in this case the differential tangent space (for example) at a good point is defined by an iterative Hasse linear differential system: $\{\partial_n(y) = A(y) : n = 1, 2, ...\}$, whose solution set is a finite-dimensional vector space over $\mathcal{C}$.

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- This turns out to be the case for $G^\sharp = p^\infty G(\mathcal{U})$ where G is an ordinary semiabelian variety (namely the p-torsion of the abelian part is maximal possible). So this yields another proof of ML in positive characteristic when the ambient semiabelian variety is ordinary.

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- This turns out to be the case for $G^\sharp = p^\infty G(\mathcal{U})$ where G is an ordinary semiabelian variety (namely the p-torsion of the abelian part is maximal possible). So this yields another proof of ML in positive characteristic when the ambient semiabelian variety is ordinary.
- It remains open to find a transparent jet-space account of Theorem 0.1 and/or Theorem 0.3 for traceless abelian varieties A in the positive characteristic case. See later.

Miscellanea

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- Moosa and Scanlon have substantially generalized the jet space arguments to fields with operators.
- Also Theorem 0.3 was used (together with other ingredients) to obtain an Ax-Lindemann theorem for nonconstant semiabelian varieties. (BP) The work is ongoing.

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- In so far as it works it also gives deductions from MM of Theorem 0.3 for example.
- Note that such an elementary strategy could not work in the absolute case, where MM and ML are of different orders of difficulty.

MM implies ML II

- It is convenient to take the contrapositive of the contrapositive in the statement of ML, with a slightly stronger hypothesis and conclusion:
- **Function field ML: restatement** Let $K = \mathbb{C}(t)^{alg}$ in char. 0, and $= \mathbb{F}_p(t)^{sep}$ in char. p and k be the "constants", $\mathbb{C}$, $\mathbb{F}_p^{alg}$ respectively. Let A be an abelian variety over K with k-trace 0. Let X be an irreducible subvariety of G (defined over K), $\Gamma \subset G(K)$ be as before, namely (prime-to-p) division points of a finitely generated subgroup of G, and assume $X \cap \Gamma$ is Zariski-dense in X. THEN X is a translate of an abelian subgroup of G (by a point of Γ). Now the MM statement is when Γ is contained in the group of all torsion points of G.

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- So the **Basic Strategy** is: MM + Theorem of the kernel + Frank implies ML (and also Theorem 0.3).

Theorem of the kernel

- $A^\sharp$ can be defined as the smallest Zariski-dense (type)-definable subgroup of A, where in the positive characteristic case we read this in a saturated model, but in any case in positive char. case $A^\sharp(K) = \cap_n p^n(A(K))$ and can also be described as the maximal divisible subgroup of $A(K)$.

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- In positive characteristic the statement was recently proved by Roessler.

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Theorem 0.5

(Frank) Suppose G is g-minimal. Then any infinite algebraically closed subset of G is an elementary substructure.

So g-minimal groups behave somewhat like strongly minimal sets. The result was originally proved by Frank for arbitrary fields of finite Morley rank, with relevance to “bad groups”.

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- By the weak socle theorem we may assume that $X^\sharp$ is contained in $s(H)$.

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- By Manin-Mumford and (*), X is a translate of an abelian subvariety of A. End of proof and/or contradiction.

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- Not always true, but I know no example of a type-definable minimal group which does not have QE.

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- By hypothesis $A^\sharp$, as a structure in its own right has QE, and hence has finite Morley rank, and is a sum of g-minimal definable subgroups.

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- By the Theorem of the kernel $A^\sharp(K)$ consists of torsion points, so by Manin-Mumford (proved by Pink-Roessler), X is a translate of an abelian subvariety of A. End of proof of Theorem 0.6.