# POINCARÉ-KOSZUL DUALITY

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Abstract. For g a dgla over a field of characteristic zero, the dual of the Hochschild homology of the universal enveloping algebra of g *completes* to the Hochschild homology of the Lie algebra cohomology of g. In this talk we will resolve this completion discrepancy through considerations of formal algebraic geometry. This will be an instance of our main result, which is a version of Poincare' duality for factorization homology as it interacts with Koszul duality in the sense of formal moduli. This can be interpreted as a duality among certain topological field theories that exchanges perturbative and non-perturbative.

## 1. Motivation

This was a chalk talk. The speaker decided not to share his hand-written lecture notes.

Joint with David Ayala.

Motivation comes from physics. They care about field theories, so in particular they care about fields (and their duals). Observables are fields$^{v}$, i.e. dualized fields.

Toy examples: start with an algebraic group $M$ and let fields be $G$-bundles on $M$, i.e. $Bun_G(M)$. Dualizing is then passage to $O(Bun_G(M))$, e.g. the sheaf of global functions.

Factorization homology began with Costello-Gwilliam, Beilinson-Drinfeld. It's supposed to be an algebraic model for observables. For simplicity let's restrict to $C = Ch^{\otimes}$. The input is an $n$-disk algebra (denoted in the previous talk by $Disk_n$-alg). The output is the chain complex of observables in this theory, denoted $\int_M A$.

An $n$-disk algebra has additional symmetries that $E_n$-algebras do not. However, for the examples coming from physics, these symmetries are always there.

Some examples:

0 For $n = 1$ we have $TV = \bigoplus_{i\geq 0} V^{\otimes i}$. Construct an action of $Top(1) \cong \mathbb{Z}/2 \cong \langle\sigma\rangle$ on this object. This $\sigma$ acts as the identity on $k$ and $V$ in $TV$, but it acts by swapping factors of $V^{\otimes 2}$.

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(1) Another good example is loop spaces. These have an $n$-disk functor given by the symmetric monoidal functor $C_*\Omega^n X : Disk_n \to Ch$ given by $C_*(Map_c(-, X))$. So the extra symmetry was always there in $Map_c(-, X)$ all-along.

(2) Commutative algebras provide a class of examples via $A \circ \pi_0 : Disk_n \to Fin \to Ch$. If you want to augment this then you use $Fin_*$.

(3) Enveloping algebras of Lie algebras (of which the first example is a special case). Given a Lie algebra $\mathfrak{g}$, we can form $C_*^{Lie}(\Omega^n \mathfrak{g})$ where $\Omega^n \mathfrak{g} \simeq \mathfrak{g}[-n]$. For $n = 1$, $C_*(\Omega\mathfrak{g}) \simeq U$. When $n > 1$ this object has a commutative monoid structure given by $\mathrm{Sym}(\mathfrak{g}[1 - n])$.

(4) Deformations of all the above also provide examples, i.e. deformations as $E_n$-algebras. This is analogous to deformation quantization.

The toy example $\mathcal{O}(Bun_G(M))$ is not equivalent to factorization homology with coefficients in any of the above. Let's quickly list off factorization homologies of the above (but not the deformations of them).

(1) $\int_M C_*\Omega^n X = C_* Map_c(M, X)$ by non-abelian Poincare Duality

(2) $\int_M A \simeq M \otimes A$ where the tensor is taken as a commutative algebra. This as discussed in David Ben-Zvi's talk.

(3) $\int_M C_*^{Lie}\Omega^n \mathfrak{g} \simeq C_*^{Lie} C_c^*(M, \mathfrak{g})$

None of these give the functor $M \mapsto \mathcal{O}(Bun_G(M))$, because it's functors on a limit rather than on a colimit. The issue is that $Bun_G(M)$ does not satisfy the local-to-global principle because passage to $Bun_G(M)$ only remembers $M$ as a space, and not the algebraic structure of $M$.

If we adjust our definition of factorization homology to allow non-affine coefficients then we can better understand this example.

## 2. Moduli Theoretic Extension of Factorization Homology

Consider the category of $n$-disk algebras in $Ch$. There is a functor $X$ to spaces, e.g. $Bun_G$. This is the functor of points approach.

$$Alg_{Disk_n}(Ch) \xrightarrow{X} \text{Spaces}$$

$$\begin{array}{ccc} Alg_{Disk_n}(Ch) & \xleftrightarrow{\text{Spec}} & Fun(Alg^{com}_{Disk_n}, Sp)^{op} \\ \Big\downarrow{\scriptstyle \int_M} & \swarrow{\scriptstyle RKE} & \\ Ch & & \end{array}$$

We may then construct

$$\int_M X = \lim_{SpecA \to X} \int_M A$$

where the limit is over all $A \in Alg_{Disk_n}/X$ and $X(A)$ is in Spec $A$.

We may now realize $O(Bun_G M)$ as $\int_M BG = \Gamma(X, \int_M O)$ where $BG$ represents $Bun_G(-)$. This realization can be done locally and then glued up to a global one using the scheme or stack structure.

**Theorem 2.1.** *Under the conditions from the previous talk (presentable ∞-category well-behaved with respect to sifted colimits):*

$$\begin{array}{ccc} int_M A & \longrightarrow & \int^{M_*^{\neg}} Bar^n A \\ \big\downarrow & \nearrow_{\simeq} & \\ P_\infty \int_{M_*} A & & \end{array}$$

When is the vertical map an equivalence? Well, we can work out one example right away.

Let $A = \mathbb{F}V \simeq \bigoplus Conf_i^{fr}(\mathbb{R}^n_+) \otimes_{\Sigma_i \wr Top(n)} V^{\otimes i}$. Then we can compute both sides of the map in question.

(1) $\int_{M_*} \mathbb{F}V = \bigoplus Conf_i^{fr}(M_*) \otimes_{\Sigma_i \wr Top(n)} V^{\otimes i}$

(2) $P_\infty \int_{M_*} \mathbb{F}V \cong \prod Conf_i^{fr}(M_*) \otimes_{\Sigma_i \wr Top(n)} V^{\otimes i}$

**Corollary 2.2.** *If $V$ is connected then this map $\int_{M_*} \mathbb{F}V \to P_\infty \int_{M_*} \mathbb{F}V$ is an equivalence.*

*Similarly, if the augmentation ideal $\overline{A}$ is connected then $\int_{M_*} A \to P_\infty \int_{M_*} A$ is an equivalence.*

However, when $A$ is not connected this map need not be an equivalence. Connectivity is determined by the fiber of the augmentation.

We really want this map to be an equivalence so that we understand what's going on (especially the filtration). One way to force this map to be an equivalence is to apply completion everywhere. But that's really an extreme measure. In the next section a more careful approach is taken.

## 3.

Features of Poincaré/Koszul Duality:

(1) The Poincaré/Koszul Duality map is not an equivalence generically for non-connected $A$, i.e. it has stuff in negative homology degrees.

(2) There exists a moduli-theoretic generalization of factorization homology, which was necessary to account for the toy example. Taking cohomology accounts for the negative homology of the dual.

Idea: given $A$, is there a moduli-functor of $n$-disk algebras related to the dual of $A$ to account for the connection between these two features?

Define $\mathbb{D}A = (\int_{(\mathbb{R}^n)^+} A)^v$. Then we are asking if there is some $X$ such that $\mathcal{O}(X) = \mathbb{D}A$.

The answer is yes, thanks to the Maurer-Cartan functor. The following definition is considered by Quillen (under the name 'twisting functions').

**Definition 3.1.** Let $\mathfrak{g}$ be a Lie algebra. Define $MC_{\mathfrak{g}}$ : Artin → Spaces via $MC_{\mathfrak{g}}(R) = Map_{Lie}(T_*R[-1], \mathfrak{g})$.

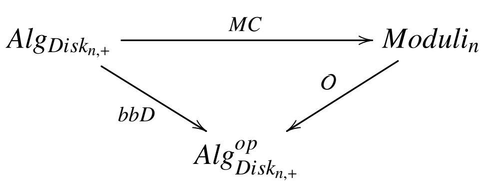
More general definitions are considered by Lurie (for $E_n$-algebras), Deligne, Kontsevich, and Drinfeld. We must first fix some notation.

Let $k$ be a field. Let $\text{Artin}_n \subset Alg^{conn}_{Disk_{n,+}}/k$ be generated by finite sequences of square zero extensions by finite $Top(n)$-modules.

Remark: This is motivated classically by taking finitely many extensions of $k[\epsilon]/(\epsilon^2)$.

**Definition 3.2.** Consider $A \in Disk_{n,+}$-alg. Define $MC_A : \text{Artin}_n \to \text{Spaces}$ by $R \mapsto Map(\mathbb{D}R, A)$.

**Lemma 3.3.** *Global functions does indeed give the Koszul dual:* $\mathcal{O}(MC_A) = \mathbb{D}A$.




Here $\mathcal{O}(X) = \lim R$ over all $R \in \text{Artin}_n/X$. This is a homotopy limit. Deriving it recovers the usual definition of $\mathcal{O}(X)$. Similarly one can define $\int_{M_*} X$ as a homotopy limit of $\int_{M_*} R$, or (equivalently) $\int_{M_*} X$ may be defined locally as $\Gamma(X, \int_{M_*} \mathcal{O})$.

**Theorem 3.4** (AF)**.** *Let $A$ be an augmented $n$-disk algebra over some field $k$. Let $M$ be a zero pointed $n$-manifold. Then*

$$\left(\int_{M_*} A\right)^v \simeq \int_{M_*^\neg} MC_A$$

Punchline: the failure in the right-hand side of the moduli problem $MC_A$ to be affine sits inside the failure of the Goodwillie tower to converge in the left-hand side.

Example: Let $C$ be an $n$-disk coalgebra such that $C^v$ is an $n$-disk algebra. Then $\int^M C \simeq (\int_{M_*} C^v)^v$.

## 4. A word about the proof of this theorem

Sadly, when you do $\mathbb{D}$ twice you don't always get back to where you started, but sometimes you do.

The essential ingredient is that the dual $\mathbb{D}A$ of an Artin algebra (i.e. an object of $\text{Artin}_n^{op}$) gives you a finitely presented $n$-disk algebra over $A$ satisfying $-n$ coconnectivity. Applying $\mathbb{D}$ twice does return you to where you started, i.e.

$$\mathbb{D} : \text{Artin}_n^{op} \leftrightarrow^{\simeq} FPres_n^{\leq -n} : \mathbb{D}$$

This equivalence is proven by a computation for $\mathbb{F}V$ where $V$ is $-n$-coconnected:

$$\int_{M_*} \mathbb{F}V \simeq P_\infty \int_{M_*} A$$

where the left hand side has layers $Conf_i^{fr}(M_*) \otimes \Sigma_i \wr Top(n) V^{\otimes i}$. These satisfy connectivity for $n + (n-1)(i-2) - in$. Now take the limit as $n \to \infty$ and you have the connectivity required to conclude the statement.

So now $(\int_{M_*} R)^v \simeq \int_{M_*^\neg} \mathbb{D}R$. It is now easy to see where this fails in general.

We use this statement to prove the main theorem. Resolve $A$ by finitely presented $n$-coconnective free algebras and apply formalism.

## 5. A Concrete Example

Let $n = 1$ and let $A = U\mathfrak{g}$ be the universal enveloping algebra for a Lie algebra $\mathfrak{g}$. When $\mathfrak{g}$ is connected:

$$\left(\int_{S^1} U\mathfrak{g}\right)^v \simeq (HC_* U\mathfrak{g})^v$$

The reason is that when $\mathfrak{g}$ is connected (i.e. there is nothing in degree 0) the left hand side can be realized as $\int_{S^1} C^*\mathfrak{g}$, which is equivalent to $HC_* C^*\mathfrak{g}$.

Very important examples (e.g. $SL(\mathbb{Z})$) are excluded by the connectivity hypothesis.

The work in this talk lets you still say something when $\mathfrak{g}$ is not connected, i.e. you can say

$$\left(\int_{S^1} U\mathfrak{g}\right)^v \simeq \left(\int_{S^1} MC\mathfrak{g}\right) \simeq HC_*(MC\mathfrak{g}$$

Another example:

$A = C_*\Omega X$

$\int_{S^1} C_*\Omega X = C_*LX$ when $X$ is connected.

When $X$ is simply connected you can plug chains-on-$X$ into the main theorem and conclude that Goodwillie calculus gives you an explicit formula for the factorization homology.