POINCARÉ DUALITY AND FORMAL MODULI

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Abstract. What is Poincaré duality for factorization homology? Our answer has three ingredients: Koszul duality, zero-pointed manifolds, and Goodwillie calculus. We introduce zero-pointed manifolds so as to construct a Poincaré duality map from factorization homology to factorization cohomology; this cohomology theory has coefficients the Koszul dual coalgebra. Goodwillie calculus is used to prove this Poincaré/Koszul duality when the coefficient algebra is connected. The key technical step is that Goodwillie calculus is Koszul dual to Goodwillie-Weiss calculus.

This was a chalk talk. The speaker's lecture notes can be found at the bottom.

Joint with John Francis. This is duality for factorization homology. It's akin to topological chiral homology (Lurie), blob homology (Morrison-Walker), and higher Hochschild homology (Dundas et. al)

1. Motivation

1.1. **Factorization homology.** Let M be an n-manifold. Let A be a Disk_n-algbra in $Ch^{\otimes}, Spec^{\wedge}$, etc.

Then factorization homology takes A to a symbol $\int_M A \in Ch$ or $Spec$ or wherever A lives.

Examples: any generalized homology, HH_*, THH

Unlike the examples above, $\int_{\_} A$ is not homotopy invariant. It still satisfies a version of excision called $\otimes$-excision, which is a local-to-global expression.

In the next talk you'll see how to interpret this symbol in terms of physics.

Take A to be a connective E_n-ring spectrum. Can look at perfect A-modules Perf_A and equivalences between them. That's a category and we can take its classifying space $B\text{Perf}_A^{equiv}$. There is a 'pre-trace' map from this space to the spectrum $\int_M A$ for all framed manifolds. What's meant by pre-trace is that in the category of A-modules it's generated by finitely many elements.

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1.2. **Poincaré Duality for Factorization homology.** Let's work in $Ch_{\mathbb{C}}^{\otimes}$. Then for M some closed manifold we can look at the linear dual of factorization homology. There is then a canonical comparison map

$$\left(\int_M A\right)^v \to \int_M \mathbb{D}^n(A)$$

which can be realized as some kind of completion at the augmentation ideal, where $\mathbb{D}$ is Koszul dual. We'll need to introduce this $\mathbb{D}$ and explain the map above.

2.

Consider the category $ZMfld_n$ of zero-pointed manifolds. The objects are locally compact Hausdorff pointed spaces M_* such that the complement of the basepoint $M = M_* - *$ is a topological n-manifold. The morphisms are maps $f : M_* \to M'_*$ which are based continuous maps such that f's restriction $f^{-1}(M') \to M'_*$ is an open embedding.

The category is pointed via the manifold which is just one point (i.e. this object is initial and terminal). Hence we have an augmentation functor in the usual way. $ZMfld_n$ is a topological category via the compact open topology on mapping spaces. The wedge sum maps it a symmetric monoidal topological category. We will regard $ZMfld_n$ as a symmetric monoidal ∞-category. Everywhere from now on that we say category we mean ∞-category.

There are subcategories

$Mfld_{n,+} \subset ZMfld_n$ given as the full subcategory of manifolds with a disjoint basepoint.

$Disk_{n,+} \subset Mfld_{n,+}$ given as full subcategory of manifolds which are finite unions of $\mathbb{R}^n_+$'s.

$Mfld_n^+ \subset ZMfld_n$ given as full subcategory of one-point compactifications of n-manifolds.

$Disk_n^+ \subset Mfld_{n,+}$ as one-point compactification of finite unions of $(\mathbb{R}^n)^+$'s.

There are operations

For any $\overline{M}$ a compact n-manifold with ∂ we get $\overline{M} \coprod_{\partial\overline{M}} *$

There is an equivalence $\neg : ZMfld_n \cong ZMfld_n^{op} : \neg$ via $(M_*)^+ - *$

There is an equivalence $Fun^{\otimes}(Mfld_{n,+}, C^{\otimes}) \to Fun^{\otimes,aug}(Mfld_n, C^{\otimes})$ for any symmetric monoidal functor $C^{\otimes}$.

Define $Alg_{Dist_n}(C)$ to be $Fun^{\otimes}(Disk_n, C^{\otimes})$

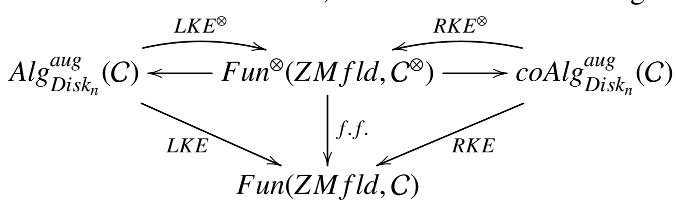
Example: $\mathrm{Alg}^{aug}_{Disk_n}(C) \to Alg_{E_n}(C)^{hTop(n)}$ is an equivalence.

Definition 2.1. For any M_* in $ZMfld_n$ and any augmented $Disk_n$-algebra A in $C^{\otimes}$ and any augmented $Disk_n$-coalgebra C we may define

factorization homology $\int_{M_*} A$ is defined to be colim($Disk_{n,+}/M_* \to Disk_{n,+} \to C$) where the last map is coming from A.

and **factorization cohomology** $\int_{M^*} A$ is defined to be colim($Disk_{n,+}/M_* \to Disk_{n,+} \to C$) where the last map is coming from A.

With this definition in mind, we can form the following functors:



here RKE and LKE are for right (left) Kan extension, and RKE$^{\otimes}$ and LKE$^{\otimes}$ are for functors which respect $\otimes$.

Theorem 2.2 (AF). *(1) If C has sifted colimits (resp. cosifted limits) then LKE (resp RKE) exists and are factorization homology (resp cohomology) functors.*

(2) If $\otimes$ distributes over the types of colimits (resp limits) above then $LKE^{\otimes}$ (resp $RKE^{\otimes}$) exist.

Note: $LKE^{\otimes}$ often exists, e.g. in chain complexes over a field. But $RKE^{\otimes}$ is more subtle.

Example: Let's work in $Ch^{\otimes}_{\mathbb{Q}}$. Then for any dg-algebra A,

$$\int_{S^3} A \simeq HH_*A \otimes_{HH_*(HH_*A)} HH_*(A)$$

Example: Let's work in spaces and let M be compact and framed. Then $\int^M C \simeq Map(M, C)$ the mapping space into the underlying space of C.

Inside of the latter example we can consider M which are finite wedge sums of $(\mathbb{R}^n)^+$. So $\int_M(A)$ is a $Disk_n$-coalgebra for such M. A complicated point-set topology fact states that this coalgebra agrees with the n-fold Bar. Furthermore, $\mathrm{Bar}^n : Alg^{aug}_{Disk_n}(C) \to coAlg^{aug}_{Disk_n}(C)$ is left adjoint to Koszul duality.

Example: $BarA \simeq 1 \otimes_A 1$. This is a geometric way of implementing the formula for $\int_M(A)$.

Summarizing:

We've defined factorization homology and cohomology. Both are formal ways of extending from the easy case of $M = \vee(\mathbb{R}^n)^+$. These respect the structure encoded by the operads $Disk_n$ and E_n.

Formally, for A some augmented $Disk_n$-algebra in $C^{\otimes}$, we may construct

$$\int_{M_*} A \to \int^{M_*^{\neg}} Bar^n(A)$$

and we don't need to make any choices.

The left-hand side can be realized as a configuration space (via Dold-Kan). The right-hand side can be realized as a mapping space. The map between them can be realized as the scanning map.

Theorem 2.3 (AF)**.** *If $C^{\otimes}$ is a topos with Cartesian product (denoted $\mathcal{X}^{\times}$) or is a stable presentable category with direct sum (denoted $\mathcal{S}^{\oplus}$) then (1) and (2) of the previous theorem are satisfied and Poincaré Duality is an equivalence provided by the map $A \to CoBar(Bar(A))$.*

Atiyah Duality is a corollary with $\mathcal{S}$ taken to be spectra.

Non-abelian Poincaré Duality (Lurie, Salvatore) is a corollary with C taken to be spaces, i.e. the ∞-category of Kan-complexes.

Example: $C^{\otimes}$ is $Ch_R^{\otimes}$. Then the above theorem does not apply. We calculate a free algebra. Let V be a Top(n)-module in C. These are the ingredients for computing $\mathbb{F}V$ the free augmented $Disk_n$-algebra and its factorization homology

$$\int_{M_*} \mathbb{F}V \simeq \bigoplus_{i \geq 0} Conf_i^{fr}(M_*) \otimes_{\Sigma_i \wr Top(n)} V^{\otimes i}$$

Note: Configurations in M_* is a zero-pointed manifold together with the information of what to do as any of the points of M are approaching the point $*$ of M_*.

So now we're looking at the filtration for the quotient and a natural question asks whether the same filtration can be used on $\int^{M_*^{\neg}}$.

3. Goodwillie-Weiss filtration and consequences

Let $Disk_{n,+}^{\leq i}$ denote the full subcategory generated as with $Disk_{n,+}$ but with at most i-many disks in M.

We get a map $Disk_{n,+}^{\leq i}/M_* \to Disk_{n,+}/M_*$. We can then stitch these together to get a truncation tower

$$\cdots \to \tau^{\leq i-1} \int_{M_*} \to \tau^{\leq i} \int_{M_*} \to \ldots$$

and the colimit of this tower is weakly equivalent to $\int_{M_*}$. Likewise we can recover $\int^{M_*}$ as the limit of a tower of $\tau^{\leq i} \int^{M}$. Furthermore, we get a fiber sequence

$$Map^{\Sigma_i \wr Top(n)}(Conf_i^{fr}(M_*)^{\neg}, C^{\otimes i}) \to \tau^{\leq i} \int^{M_*} C \to \tau^{\leq i-1} \int^{M_*} C$$

Comparing layers yields

$$Conf_i^{fr}(M_*) \otimes_{\Sigma_i \wr Top(n)} V^{\otimes i} \to Map^{\Sigma_i \wr Top(n)}(Conf_i^{fr}(M_*)^{\neg}, ((\mathbb{R}^n)^+ \otimes V)^{\otimes i}$$

This map is a weak equivalence by our earlier Atiyah duality corollary to Poincaré Duality.

Theorem 3.1 (AF)**.** *The Poincaré Duality map induces an equivalence of towers from* $P_\bullet \int_{M_*}$*, the Goodwillie tower of the functor* $\int_{M_*} : Alg_{Disk}^{aug}(C) \to C$*, to the Goodwillie-Weiss tower* $\tau^{\leq \bullet} \int^{M_*^{\neg}} Bar(-)$*.*

As a corollary we get Poincaré Duality: $\int_{M_*} A \to \int^{M_*^{\neg}} Bar(A)$ and it factors through $P_\infty \int_{M_*} A$.

Some questions: what sort of analyticity do these maps have? Why don't we get Poincaré Duality in general? Is it related to the failure of the Goodwillie tower to converge?

Joint w/ John Francis

— Thanks —

MSRI
April 2014

"Factorization Homology" (Akin to "Top chir Hmlgy" (Lurie), "Blob Hmlgy" (Morrison-Walker), "Higher Hochschild" ()

What is it?

M, $A \longmapsto \int_M A \in Ch_{\mathbb{k}}$, spec, ...

n-mfld, algebra in $Ch_{\mathbb{k}}^{\otimes}$, $Spec^{\wedge}$, ... (over $Disk_n$-operad)

Facts

- (Generalized) Hmlgy, and HH_* & THH, give examples.
- $\int A$ is not a htpy invariant yet satisfies a local-to-global expression "$\otimes$-excision".
- $\int_M A$ can be interp as the global obsvbls of a perturb TQFT (Costello-Gwilliam)
- A connective E_n-Ring Spec $\rightsquigarrow B(Perf_A^{\geq 0, equiv}) \xrightarrow{\text{"Pre-Trace"}} \int_M A \;\; \forall M$ framed

Question What is Poincaré duality for Fact Hmlgy?

$$\left(\int_M A\right)^{\vee} \xrightarrow[\text{@ aug}]{\text{cpHn}} \int_M \mathbb{D}^n A$$

(linear dual; M: cpt fr n-mfld; A: aug E_n-alg; $\mathbb{D}^n A$: Koszul dual) (in $Ch_{\mathbb{k}}^{\otimes}$)

Campbell

Def $ZMfld_n$ is a category w/

ob: M_* based loc cpt Haus top spcs s.t. $M := M_* \setminus *$ is a topological n-mfld.

mor: $M_* \xrightarrow{f} M'_*$ based map s.t. $f^{-1}M' \to M$ is an open embedding.

- Cpt-open toplgy } Sym mon toplgcl cat.
- $\vee$ wedge

Sub-sym mon top cats

$$Mfld_{n,+} \subset Mfld_{n,+}^{+} \supset Mfld_n^{+}$$

($\{M_+\}$ disj pt; $\{M_+\} \cup \{M^+\}$; $\{M^+\}$ one-pt cptfn)

$$\cup \qquad \cup \qquad \cup$$

$$Disk_{n,+} \qquad Disk_{n,+}^{+} \supset Disk_n^{+}$$

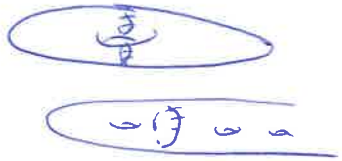
$\{V_{fin}(\mathbb{R}^n)^+\}$

$\mathcal{Z}\mathrm{Mfld}^{fin} \subset \mathcal{Z}\mathrm{Mfld}$

Smallest s.t. $\mathbb{R}_+^{n-k} \wedge (\mathbb{R}_{\geq 0}/_{\{0\}})^{\wedge k}$ is finitary

& closed under "collar-gluings"

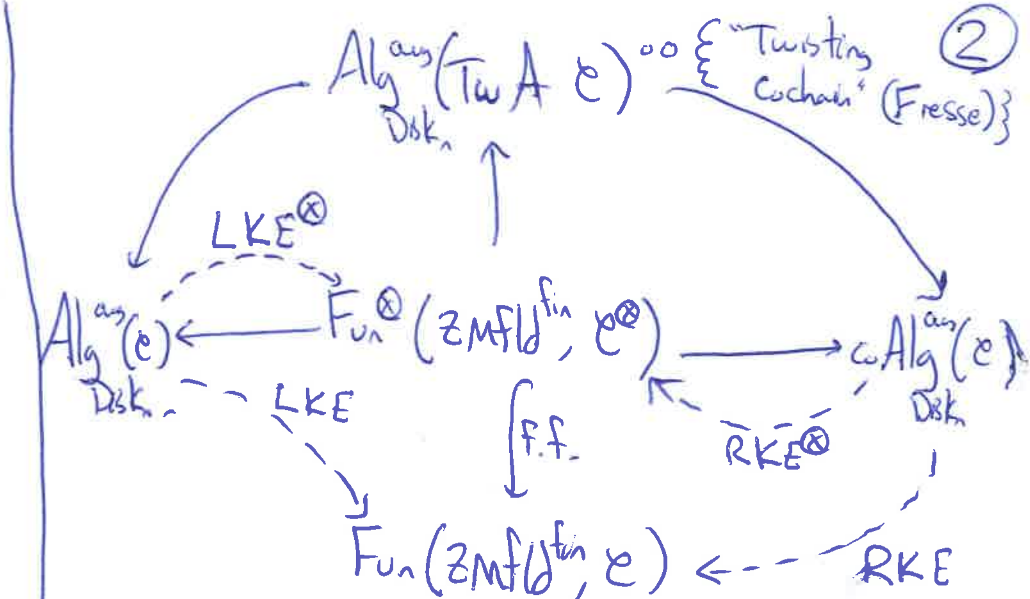
$(M_{Left})_* \coprod_{\mathbb{R}_+ \wedge \partial_*} (M_R)_* \cong M_*$



Facts

- $(\overline{M}, \partial\overline{M} = \partial_L \amalg \partial_R)$ a toplgcl cobord
  $\Rightarrow (\overline{M} \setminus \partial_R) \coprod_{\partial_L} *$ is finitary (Quinn, Kirby-Siebenmann, Moise)
- $\neg : \mathcal{Z}\mathrm{Mfld} \cong \mathcal{Z}\mathrm{Mfld}^{op} : \neg$ "negation"
  $\mathrm{Mfld}_{n,+} \hookrightarrow$; $\mathrm{Mfld}_{n,+} \cong (\mathrm{Mfld}_n^+)^{op}$
- $\mathrm{Alg}^{aug}_{\mathrm{Disk}_n}(\mathcal{C}) \cong \mathrm{Fun}^{\otimes}(\mathrm{Disk}_{n,+}, \mathcal{C}^{\otimes}) \xrightarrow[(K\text{-}S)]{\simeq} \mathrm{Alg}^{aug}_{E_n}(\mathcal{C})$ — Smoothing Thry $n \neq 4$; $\mathrm{Alg}_{A_\infty}(\mathrm{Alg}_{A_\infty}(\cdots(\mathcal{C})\cdots))$, n; $h\mathrm{Top}(n)$
  (Kontsevich, Lambrechts-Volić) $\mathrm{Alg}^{aug}_{\mathrm{Poisson}_n}(\mathcal{C})$ $n \geq 2$, $\mathcal{C}^{\otimes} = \mathrm{ch}^{\otimes}_{\mathbb{Q}}$; $\mathrm{Top}(n)$ (Kister-Mazur)
- (harder)

$\mathrm{Alg}^{aug}_{\mathrm{Disk}_n}(\mathrm{Tw}\,\mathcal{A}\,\mathcal{C}) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Disk}^+_{n,+}, \mathcal{C}^{\otimes})$



Fact

① Spose $\mathcal{C}$ admits sifted colims (coSftd lims)
Then LKE (RKE) exists.

② Spose ① & $\otimes$ distributes over.
Then LKE$^{\otimes}$ (RKE$^{\otimes}$) exists AND is fully fth
AND essential image is the $\otimes$-excisive H $\left[H(M_L)_* \otimes_{H\,\mathbb{R}_+ \wedge \partial_*} H(M_R)_* \xrightarrow{\simeq} H M_* \right]$

Def M_* a finitary $\mathcal{Z}$-pmfld. A (C) an aug Disk_n-alg (coalg)

$\int_{M_*} A := \mathrm{LKE}(A)|_{M_*} \simeq \mathrm{Colim}(\mathrm{Disk}_{n,+/M_*} \to \mathrm{Disk}_{n,+} \xrightarrow{A} \mathcal{C})$

$\int^{M_*^{\neg}} C := \mathrm{RKE}(C)|_{M_*^{\neg}} \simeq \lim(\mathrm{Disk}^+_{n\,M_*^{\neg}/} \to \mathrm{Disk}^+_n \xrightarrow{C} \mathcal{C})$

Ex • $\int_{S^3} A \simeq HH_*A \otimes HH_*A$
$\uparrow$ $HH_*(HH_*A)$

$\mathcal{C}^\otimes = Ch_{\mathbb{Q}}^\otimes$

• $\int^{M_*} C \simeq Map^{Top(n)}(M_*^{fr}, C)$

$\mathcal{C}^\otimes = Spaces^\times$

Fact | $Bar^n : Alg^{aug}_{Disk_n}(\mathcal{C}) \rightleftarrows coAlg^{aug}_{Disk_n}(\mathcal{C}) : coBar^n$ ($\perp$; $\int_-$ just on $\bigvee_{\text{finite}} \mathbb{R}^n_+$; $\int^-$)

Is the familiar Koszul duality adjunction.

Formally, $A \in Alg^{aug}_{Disk_n}(TwAr\,\mathcal{C})$ w/ $A_+ := A|_{Disk_{n,+}}$, $A^+ := A|_{Disk_n^+}$

$\rightsquigarrow \int_{M_*} A_+ \xrightarrow{PD_A} \int^{M_*^\neg} A^+$ {plus "twisting cochain" (Fresse)}

{ $Sp^\infty(M;A) \xrightarrow{\text{"Scanning"}} Map(M, B^nA)$ (Segal, Bödigheimer) }

Thm (AF) Spose $\mathcal{C}^\otimes = \mathcal{X}^\times$ Topos or $\mathcal{S}^\oplus$ Stable presentbl. Then PD_A is an $\simeq$ provided $A_+ \xrightarrow{\simeq} coBar^n A^+$ / $Bar^n A_+ \xrightarrow{\simeq} A^+$.

$\mathcal{C}^\otimes = ch_{\mathbb{Z}}^\otimes$? (Lurie; Salvatoré, Atiyah)

(Co)Filtrations

Fix M_* finitary z-p n-mfld and $\mathcal{C}^\otimes$ w/ $\mathcal{C}$ stbl pres & seq colims commutes w/ lims & $\otimes$ distribute over colims.

Goodwillie-Weiss Filtration

$Disk^{\le i}_{n,+}/M_* \longrightarrow Disk_{n,+}/M_*$

subcat of those $\bigvee_I \mathbb{R}^n_+ \xrightarrow{f} M_*$ s.t. $Card(f(\sqcup_I \{0\}) \cap M) \le i$

&

$Disk^{+,\le i}_{n\,M_*/} \longrightarrow Disk^+_{n\,M_*/}$ likewise

$Colim(\cdots \to \tau^{\le i-1}\int_{M_*} \to \tau^{\le i}\int_{M_*} \to \cdots) \xrightarrow{\simeq} \int_{M_*}$

$\rightsquigarrow$ &

$\int^{M_*} \xrightarrow{\simeq} \lim(\cdots \to \tau^{\le i}\int^{M_*} \to \tau^{\le i-1}\int^{M_*} \to \cdots)$

Goodwillie Filtration

$\int_{M_*} : Alg^{aug}_{Disk_n}(\mathcal{C}) \longrightarrow \mathcal{C}$

$\rightsquigarrow \int_{M_*} \to \cdots \to P_i\int_{M_*} \to P_{i-1}\int_{M_*} \to \cdots$

Thm (AF) The PD map induces an equivalence of towers

$$P_\bullet \int_{M_*} A \xrightarrow{\ \simeq\ } \tau^{\le \bullet} \int^{M_*^\neg} \mathrm{Bar}^n A .$$

In particular

$$\int_{M_*} A \xrightarrow{\ PD\ } \int^{M_*^\neg} \mathrm{Bar}^n A$$

$$\int_{M_*} A \xrightarrow{\ J\ } P_\infty \int_{M_*} A \xrightarrow{\ \simeq\ } \int^{M_*^\neg} \mathrm{Bar}^n A .$$

Sketch

$$\mathrm{Conf}_i^{fr}(M_*) \underset{\Sigma_i \wr \mathrm{Top}(n)}{\otimes} (LA)^{\otimes i} \xrightarrow[\simeq]{\ (3)\ } \mathrm{Map}^{\Sigma_i \wr \mathrm{Top}(n)}\left(\mathrm{Conf}_i^{fr}(M_*^\neg), \mathrm{Bar}^n A\right)$$

$$\downarrow (1) \qquad\qquad\qquad\qquad \downarrow (2)$$

$$P_i \int_{M_*} A \dashrightarrow \tau^{\le i} \int^{M_*^\neg} \mathrm{Bar}^n A$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$P_{i-1} \int_{M_*} A \xrightarrow[\text{Induction}]{} \tau^{\le i-1} \int^{M_*^\neg} \mathrm{Bar}^n A$$

(1) Classif'n of Homog Fctrs ⇒ Just check for A free.

Calculate $\int_{M_*} \mathbb{F} V \simeq \bigoplus_{i \ge 0} \mathrm{Conf}_i^{fr}(M_*) \underset{\Sigma_i \wr \mathrm{Top}(n)}{\otimes} V^{\otimes i}$

use that $L \mathbb{F} V \simeq V$

(2)

$$\left(\mathrm{Disk}^{\le i}_{n,+}/M_*\right) \Big/ \left(\mathrm{Disk}^{\le i-1}_{n,+}/M_*\right) \xrightarrow[\Lambda^i]{\ \simeq\ } \underbrace{\mathrm{Disk}^{\le 1}_{ni,+} \big/ \mathrm{Conf}_i(M_*^\neg)}_{\mathrm{Conf}_i^{fr}(M_*^\neg)} \quad (4)$$

(3) $(\mathbb{R}^n)^+ \otimes LA \underset{(\text{Francis})}{\simeq} \mathrm{Bar}^n A$

This is the PD map applied to $\mathrm{Conf}_i(M_*)$ w/ coefs in $(LA)^{\otimes i}$

Wonder

- When is $\int_{M_*} A \to P_\infty \int_{M_*} A$ an equiv?
- Conceptual Framework?

‖Alg Geom for Disk_n-alg's‖