MODELING STABLE HOMOTOPY 2-TYPES

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This was a chalk talk. The speaker's hand-written notes are at the bottom.

Joint work in progress with Nick Gurski and Niles Johnson.

Homotopy theorists have long been trying to find algebraic models for homotopy types. We are seeking a categorical model for stable 2-types.

Theorem 0.1 (Thommason's Theorem)**.** *Let Cat be the category of categories. There is a Quillen equivalence of model categories Cat $\simeq Top$ where the Thommason model structure is taken on Cat, i.e. the weak equivalences are the functors $F : C \to \mathcal{D}$ such that the map of classifying spaces $BC \to B\mathcal{D}$ is a weak equivalence.*

This means any space can be constructed up to weak equivalence as a classifying space of some category.

Theorem 0.2 (Thommason and Mandell)**.** *The category SymMonCat of symmetric monoidal categories is Quillen equivalent to connective spectra.*

We again get a recognition principal via classifying spaces.

The difficulty of modeling the full homotopy type of a space might be behind the need to consider the Thommason weak equivalences rather than simply categorical equivalences. That's our jumping off point.

1. Homotopy n-types

A *homotopy n-type* is a space X such that for all $i > n$ and all $x \in X$ $\pi_i(X, x) = 0$.

Given a space Y, the Postnikov n-truncation is a homotopy n-type. It seems exactly the lower homotopy groups.

You can get at homotopy n-types via Bousfield localization (inverting maps which are isomorphisms on $\pi_{<n}$.

A connected 1-type is a space which is connected and has only one homotopy group. So it's just $BG = K(G, 1)$ where G is $\pi_1(X)$.

A connected 2-type is a crossed module, by a result of Whitehead.

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Crossed modules of length n are supposed to model n types. A different generalization is the notion of a Cat_n-group.

We will generalize in the direction of groupoids. The *homotopy hypothesis* says that weak n-groupoids model homotopy n-types. This is a theorem for $n = 1, 2, 3$.

Defining strict n-categories is easy. You just go inductively. Weak n-categories are harder but they give the right thing for the homotopy hypothesis. Let's look at the case $n = 1$.

The functors Groupoids$\leftrightarrows Top_{\leq 1}$ is induced by functors (B, π_1) between Cat and Top. Here π_1 is for fundamental groupoid. The objects of $\pi_1(X)$ are the points of X and the morphisms are homotopy classes of maps. That you land in groupoids is just because paths can be reversed. It's a theorem that via (B, π_1) the homotopy categories of Cat and Top are equivalent. Note that (B, π_1) need not be an adjunction (because we did not necessarily pass through sSet).

Observe: equivalences in groupoids are just the categorical equivalences, because the groupoid information is contained in π_0 and π_1, so an equivalence on classifying spaces sees everything. Now we move on to $n = 2$.

A *bicategory* is the weak version of a 2-category. So you have objects, morphisms, and 2-morphisms. For any two objects X, Y there is a category $C(X, Y)$. Morphisms in this category give you the notion of 2-morphisms. There must also be composition $C(Y, Z) \times C(X, Y) \to C(X, Z)$ and a functor $1_X \in ob(C(X, X))$ satisfying $h \circ (g \circ f) \to (h \circ g) \circ f$ and natural isomorphisms $1_Y \circ f \to f$, $f \circ 1_X \to f$. The remaining axioms codify the point that the bicategory is a symmetric monoidal category with many objects. So there's an associative pentagon for instance.

Remark: there is in fact a tricategory of bicategories, but we won't go there.

Examples:

(1) If C is a category then it's a bicategory where all the 2-morphisms are identities. So $C(X, Y)$ is the discrete category on the set $C(X, Y)$.

(2) The collection of small categories is a bicategory via categories, functors, and natural transformations. In fact this is a strict 2-category because associativity for composition of functors holds on the nose.

(3) Let R be a commutative ring. Then $Bimod_R$ is a bicategory whose objects are R-algebras, 1-morphisms $A \to B$ are A-B bimodules, and 2-morphisms are bimodule maps. The composition of $A \to B$ and $B \to C$ is given by $- \otimes_B -$. This is not associative on the nose but only up to coherent isomorphisms.

(4) Let X be a space. Consider the bigroupoid $\pi_2 X$ whose objects are points of X, 1-morphisms are paths $x \to y$, and 2-morphisms are homotopy classes of homotopies. Composition is associative up to a 2-morphism because we are not taking Moore path spaces.

In general a *bigroupoid* is a bicategory with invertible 2-morphisms and 1-morphisms are invertible up to isomorphism (making use of the 2-morphisms).

The classifying space functor $B : Bicat \to Top$ factors through $sSet$ as the nerve followed by geometric realization. Here the 0-simplices NC_0 are the objects of C, NC_1 are the 1-morphisms of C, and NC_2 are fillers for triangles from $0 \to 1 \to 2$ to the composition $0 \to 2$.

Theorem 1.1 (Moerdijk-Svensson)**.** *There is an equivalence of homotopy categories between Bigroupoids mod biequivalences and homotopy 2-types. This is given by the classifying space in one direction and the fundamental bigroupoid in the other direction.*

The notion of biequivalence is that there is a pseudofunctor in each direction and the composites are suitably equivalent to the identity functors.

Define $\pi_0(C)$ to be equivalence classes of objects, i.e. the connected components of the corresponding graph. Next, for $c \in C$ an object, $\pi_1(C, c) = C(c, c)/\text{iso}$. Finally, $\pi_2(C, c) = C(c, c)(1_c, 1_c)$. We can see this is a group. Eckmann-Hilton says it's actually an abelian group. These coincide with $\pi_i(BC, c)$.

2. Stable World

A *stable homotopy n-type* is a spectrum X such that the homotopy groups $\pi_i(X) = 0$ for $i < 0$ and $i > n$.

We will be working with grouplike E_∞ n-types. When $n = 0$ this is just the category of abelian groups. For $n = 1$ this is answered by the following folk theorem.

Theorem 2.1. *The homotopy category of grouplike symmetric monoidal groupoids (aka Picard categories) is equivalent to the category of stable homotopy 1-types. Here the equivalences are categorical equivalences.*

Now we move on to $n = 2$, stopping first to collect a different way to look at the $n = 1$ case.

Theorem 2.2 (Johnson-Osorno)**.** *The Postnikov data of a Picard category C is $\pi_0(C)$ the connected components, $\pi_1(C) = C(I, I)$, and $\pi_0(C) \to \pi_1(C)$ is given by the symmetry isomorphism $x \otimes x \to x \otimes x$, i.e. x is taken to the map $C(I, I)$ induced by this symmetry isomorphism and the fact that C is Picard (so an automorphism of any object gives an automorphism of the unit via inverting the object).*

For $n = 2$ we will need to know that Picard bicategories are grouplike symmetric monoidal bigroupoids. Let's parse this sentence.

A *symmetric monoidal bicategory* is a bicategory C together with a pseudofunctor $\otimes : C \times C \to C$, a unit object I, associators $x \otimes (y \otimes z) \to (x \otimes y) \otimes z$, twists $\tau : x \otimes y \to y \otimes x$, and unit maps $I \otimes x \to x \leftarrow x \otimes I$ all pseudonatural equivalences subject to certain diagrams which commute up to a 2-morphism between the two ways of going around. Then there are pages and pages encoding about 7 axioms.

The key piece that distinguishes a braided monoidal category from a symmetric monoidal category is that in the latter has an extra condition regarding $\tau \circ \tau$ vs. the identity. In fact there are other conditions one could choose. One choice leads to sylleptic categories.

Example: $Bimod_R$ with monoidal structure given by $A \otimes_R B$.

Theorem 2.3 (Gurski-Osorno)**.** *These data are coherent. The classifying space BC is an E_∞-space.*

This means you can actually construct a spectrum when you group complete.

Proof sketch: mimic Segal's proof. Start with a symmetric monoidal bicategory C and construct a special Γ-bicategory $\Gamma^{op} \to Bicat$. Then construct a special Γ-space $\Gamma^{op} \to Top$. Your choices don't matter thanks to the coherence result.

Theorem 2.4 (Gurski-Johnson-Osorno, 80% proven)**.** *There is an equivalence of homotopy categories between Picard bicategories mod categorical equivalence and stable homotopy 2-types.*

4/8/14

Modeling stable 2-types (joint w/ N. Gurski & N. Johnson)

MSRI Conference

Homotopy theorists have long been concerned w/ finding algebraic models for homotopy types. Talk today about categorical models.

Thomason's results (on one hand)

$$\mathcal{C}at \underset{Q}{\simeq} Top$$

$\mathcal{C} \to \mathcal{D}$ w. equiv if $B\mathcal{C} \to B\mathcal{D}$ is w. equiv

This reads that all homotopy types can be obtained as the classifying space of a category.

Analogous,

$$SymMon\,\mathcal{C}at \simeq Spectra_{\geq 0}$$

(connective) (Thomason, Mandell)

Unsatisfactory: hard to understand w. equiv & corresponding category to a homotopy type.

In a different direction, homotopy n-types

Def A space X is a homotopy n-type if for all $x \in X$ and $i > n$,

$$\pi_i(X, x) = 0.$$

(can alternatively say that the homotopy n-type of a space is its n-th Postnikov truncation)

There has been a lot of work in classifying n-types algebraically.

Connected 1-types $\leftrightarrow$ groups (via BG)

Connected 2-types $\leftrightarrow$ crossed modules (Whitehead)

~~Conduché~~ Loday : Cat-n-groups
Conduché : Generalized crossed modules.

Homotopy hypothesis : "weak n-groupoids model homotopy n-types".

For general definitions of n-category this has been verified (Tamsamani, Padi)
For concrete definitions verified for $n=1,2,3$

Hypothesis precedes the definition.

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n=1 $\mathrm{Top} \xrightarrow{\Pi_1} \mathrm{Gpd}$ fundamental groupoid

$\Pi_1 X$ ob points of X
mor htpy classes of paths.

$\mathrm{Gpd} \xrightarrow{B} \mathrm{Top}$ lands in htpy 1-types.

$B\Pi_1 X$ is postnikov truncation.

Fact $\mathcal{C} \to \mathcal{D}$ is equivalence of cats iff $B\mathcal{C} \to B\mathcal{D}$ is w. equiv.

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n=2 **Def** A bicategory $\mathcal{C}$ consists of ob $\mathcal{C}$ (set / collection...)

$\forall x,y \in \mathrm{ob}\,\mathcal{C}$, $\mathcal{C}(x,y)$ a category,

Composition $\mathcal{C}(y,z) \times \mathcal{C}(x,y) \to \mathcal{C}(x,z)$ functor

identity $1_x \in \mathrm{ob}\,\mathcal{C}(x,x)$

associative & unital up to natural isomorphisms.

Have notions of (pseudo) functors, (pseudo) natural transf.

Ex ① If $\mathcal{C}$ is a category, we can think of it as a bicategory w/ only id 2-cells.

② **Cat** categories
functors
natural transf.

(3) R commutative ring
Bimod_R ~~(R)~~ R-alg

$\mathrm{Bimod}_R(A,B)$ = A-B bimodules
bimodule maps.

Composition $\otimes_B$ unit A as an A-A bimodule.

(4) If X is a space $\Pi_2 X$ fundamental bigroupoid. — invertible 2-cells, 1-cells inv up to iso

$\Pi_2 X$ points of X
paths $x \to y$
htpy classes of homotopies.

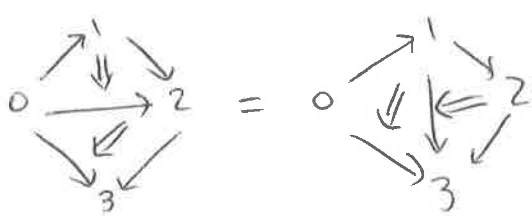
$B : \mathrm{Bicat} \xrightarrow{N} \mathrm{sSets} \xrightarrow{|-|} \mathrm{Top}$

$N\mathcal{C}_0 = \mathrm{ob}\,\mathcal{C}$

$N\mathcal{C}_1$ = 1-cells in $\mathcal{C}$

$$N\mathcal{C}_2 = \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd}$$

$N\mathcal{C}_3 =$



3-coskeletal.

<u>Thm</u> (Moerdijk-Svensson)
There is an equivalence of htpy cats
Bigroupoids / biequivalence $\simeq$ Homotopy 2-types
($B\Pi_2 X$ is a model for Postnikov piece).

given by B and Π_2.

<u>Note</u> $\mathcal{C}$ a bigroupoid

$\pi_0 \mathcal{C} = \mathrm{ob}\,\mathcal{C}/\sim$

$\pi_1(\mathcal{C}, c) = \mathcal{C}(c,c)/\cong$

$\pi_2(\mathcal{C}, c) = \mathcal{C}(c,c)(1_c, 1_c)$

$\pi_0 \mathcal{C} \cong \pi_0 B\mathcal{C}$

$\pi_i(\mathcal{C}, c) \cong \pi_i(B\mathcal{C}, c)$ $i = 1,2$

$\pi_i(B\mathcal{C}, c) = 0$ if $i > 3$.

What happens in the stable world?

Def A stable homotopy n-type is a spectrum X with $\pi_i X = 0$ for $i<0$ and $i>n$.

~~"Stable homotopy hypothesis": "group like symmetric monoidal weak n-groupoids model stable homotopy n-types."~~

~~explain why you need grouplike~~

work w/ grouplike E_∞-spaces.

$n=0$ Abelian groups HA.

$n=1$ ~~Folklore thm~~

~~Picard categories = gro~~

Groupoids $\longleftrightarrow$ unstable 1-types

Symm mon cats $\longleftrightarrow$ E_∞-spaces

grouplike $\longleftrightarrow$ grouplike

Folklore thm: Picard categories = grouplike symm mon groupoids $\simeq$ stable 1-types.

Thm (Johnson-O.)
The k-invariant $\pi_0(\mathcal{C}) \longrightarrow \pi_1(\mathcal{C})$ is induced by the symmetry of the symmetric monoidal structure on $\mathcal{C}$.

$x\otimes x \longrightarrow x \otimes x$

Stable homotopy hypothesis: "grouplike symmetric monoidal n-groupoids model stable htpy n-types".

$n=2$ Picard bicategories = grouplike symm mon bigroupoids.

Definition of symm monoidal bicategory is complicated

$\mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ pseudofunctor

$(x\otimes y)\otimes z \longrightarrow x\otimes(y\otimes z)$
$x\otimes y \longrightarrow y\otimes x$
$I\otimes x \longrightarrow x$
$x\otimes I \longrightarrow x$

pseudonat equiv.

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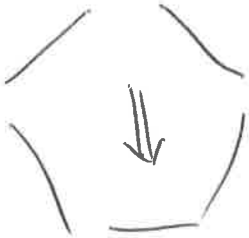

invertible 2-morph

$$\begin{tikzcd} x\otimes y \arrow[rr, equal] \arrow[dr] & \Downarrow & x \otimes y \\ & y \otimes x \arrow[ur] & \end{tikzcd}$$

+ axioms

Ex Bimod_R is sym mon $A \otimes_R B$

Thm (Gurski – O.) This data is coherent.

Thm (O., Gurski – O.) If $\mathcal{C}$ is a symm mon bicat, $B\mathcal{C}$ is an E_∞-space.

There is a spectrum $K\mathcal{C}$.

Idea of proof: Segal's Γ-spaces

$\Gamma^{op} = \mathrm{FinSet}_*$ $\quad$ $\mathcal{C}$ symm mon bicat $\rightsquigarrow$ $\mathrm{FinSet}_* \longrightarrow \mathrm{Bicat}$

$1 \longmapsto \mathcal{C}$

special Γ-bicategory

Ex $\mathrm{Az}_R \subseteq \mathrm{Bimod}_R$

↳ invertible things

Can construct spectrum $K\mathrm{Az}_R$

$\pi_0 = \mathrm{Br}(R)$

$\pi_1 = \mathrm{Pic}(R)$

$\pi_2 = R^\times$

$\pi_i = 0 \quad i > 2$

related to Szymik

Thm (Gurski – Johnson – O.) (80%) There is an equivalence of homotopy categories

$$\text{Picard bicat}/\text{equiv} \simeq \mathrm{Spectra}_0^2/\sim$$

Idea of proof:

sym mon bicat $\simeq$ Γ-bicat $\simeq$ Γ-spaces $\simeq$ $\mathrm{Spectra}_{\geq 0}$ $\quad$ a la Mandell.