THE UNICITY OF THE HOMOTOPY THEORY OF HIGHER CATEGORIES

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ABSTRACT. Higher categories are playing an increasingly important role in algebraic topology and mathematics more generally. Due to their diverse origins there are many competing approaches to the theory. In this talk I will describe joint work with Clark Barwick which gives a solution to the comparison problem in higher category theory. We give a brief axiomatization of the theory of (∞, n)-categories (and other closely related theories). From this we show that the space of homotopy theories satisfying these axioms is $B(Z/2)^n$, and hence any two theories satisfying the axioms are equivalent with very little ambiguity in *how* they are equivalent. Examples of popular theories which satisfy these axioms will be provided along with a spattering of applications.

There will be no notes from the speaker because this material is contained in the (expository) last section of the lecture notes hosted as arxiv 1308.3574.

The notes presented here should be seen mainly as a supplement to the notes above.

1. History

Mac Lane's coherence theorem tells you that you can always replace a weak 2-category by a strict 2-category. Ross and Street tried to do the same for 3-categories in 1995, and it turns out you can't. Batanin also came up with an idea for what a weak 3-category should be, as did two other independent teams of researchers.

There are strict 2-categories, opetopic 2-categories, and 2-relative categories.

Let A be a category. Then $\sigma(A)$ is a 2-category where A sits between the 0-cells 0 and 1. So $\mathrm{Hom}(0,0) = pt = \mathrm{Hom}(1,1)$, $\mathrm{Hom}(0,1) = A$, $\mathrm{Hom}(1,0) = \varnothing$.

Opetopic 2-categories have a certain horn-filling property via universal arrows.

2-relative categories are simply diagrams

$$\begin{array}{ccc} v_1C & \longrightarrow & aC \\ \uparrow & & \uparrow \\ wC & \longrightarrow & v_2C \end{array}$$

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where all objects in the diagram are categories (all on the same objects) and all maps are inclusions. Every morphism of aC is a finite composite of v_iC's. The only relations are given by composition rules $f_2 \circ g_1 = g_2 \circ f_1$ where $g_1, g_2 \in v_iC$ and $f_1, f_2 \in v_jC$.

The point is that there were many options when homotopy theorists wanted to define ∞-categories. In this talk we'll axiomatize the notion and prove that any approach which satisfies the axioms is equivalent to the other approaches. By analogy, think about the Eilenberg-Steenrod axioms.

Another analogy is given by May-Thomason's uniqueness of infinite delooping machines. They first choose to work in a single theory (Segal's theory) and then compare all the other theories to that theory via an axiomatization.

More details on May-Thomason: Given an E_∞ operad they produce a new category which we'll call $MT(E_\infty)$, whose morphism spaces are built using E_∞. This category $MT(E_\infty)$ is equivalent to Segal's category Γ, which is MT of the terminal operad. Let Π denote MT of the initial operad. It should be thought of as 'Segal maps.' If E_∞ is levelwise equivalent to the terminal operad then you'll get comparison maps between their algebras. There is a standard way to get from Γ to connective spectra, and this allows for a functor from $MT(E_\infty)$ to connective spectra. It turns out that the images of this functor for different E_∞'s in fact coincide.

The lesson we draw from this is that it's important to have a place to work where all the models live and can be compared.

2. Different choices for the homotopy theory of homotopy theories

We want our answer to have function objects of the same type, i.e. a way to ddo homotopy theory on the maps between two homotopy theories. As a result, we reject model categories because functors between two model categories need not be a model category.

RelCat is the collection of categories C with a distinguished subcategory $\mathscr{W}$ on the same objects (and containing all identity maps). Even in RelCat it's hard to formulate function objects.

Cat_Δ is the collection of simplicial categories. Hammock Localization L^H takes you from RelCat to Cat_Δ. Note that Cat_Δ contains more information than simply categories enriched in $Ho(Top)$, i.e. categories whose spaces are n-types.

CSS is complete Segal spaces (introduced by Rezk as a model for the homotopy theory of homotopy theories). The classification diagram functor goes from RelCat to CSS.

Seg^{inj} and Seg^{proj} are two model structures on the category of Segal categories (both introduced by Bergner).

There are other options too, most notably quasi-categories, which is the setting we work in to compare the others.

Recall from Angelica's talk that (groupoids, categorical equivalences)$\simeq$ 1-types. The more general version of this is the *Homotopy Hypothesis*, which states (weak n-groupoids, equivalences) $\simeq$ n-types. So letting $n \to \infty$ says $(\infty, 0)$-categories are supposed to be the same thing as spaces.

We think of weak n-groupoids as $(n + k, n)$-categories, i.e. $n + k$-categories where everything is invertible above dimension n. Note that an (n, n)-category is the same as a weak n-category, and an $(n, 0)$-category is an n-groupoid. An $(\infty, 1)$-category is an ∞-category (i.e. has n-morphisms for all n) and morphisms between any two objects form an $(\infty, 0)$-category, i.e. a space.

qCat is an implementation of $(\infty, 1)$-categories. Thanks to Joyal, Lurie, and others this is the setting with the most machinery already in place to do our comparison.

All the collections of categories discussed in this section are model categories, and all are Quillen equivalent through a zig-zag. There is a great picture in the unicity paper's introduction which contains these models and Quillen equivalences (the right adjoints only) between them.

Julie Bergner has shown that two different ways of going from Cat_Δ to CSS are equivalent up to a long zig-zag in CSS. Proving this in general is hard. In particular, it was unknown that this diagram of models and Quillen equivalences commuted. The point of this work is to prove that all these Quillen equivalences are coherent and that in particular the diagram commutes. This solves the 'monodromy problem' of taking different paths around the diagram.

Furthermore, any other reasonable model you might propose will also be equivalent to qCat. By 'reasonable model' we mean satisfying some axioms.

3. Axiomatization

There is a huge simplicial set whose vertices are quasi-categories satisfying the axioms, the 1-simplices are equivalences of quasi-categories, and whose higher simplices have explicit descriptions which we will not discuss.

Theorem 3.1. *This simplicial set is a Kan complex and is a $B(\mathbb{Z}/2)^n$.*

The $\mathbb{Z}/2$ is there because you can always take an n-category to it's opposite. Indeed, you can flip around any of the k-morphisms for $k < n$ and you get 2^n different such images of an (∞, n)-category (all of which are equivalent to the original (∞, n)-category in a strong way).

The method of proof of the unicity theorem is to introduce the category of 0-truncated objects $\tau_{\leq 0}C \subset C$. The objects X satisfy the property that $C(Y, X)$ is a discrete space. So these X are the ones without any interesting homotopy theory.

For example, $\tau_{\leq 0}Top$ is just Set (a.k.a. 0-types). Less trivially, $\tau_{\leq 0}Cat$ is the subcollection of categories where the only isomorphisms are identities. Let $Gaunt_1$ denote $\tau_{\leq 0}Cat$. To see this, consider that $Fun^0(C, \mathcal{D})$ is the 1-groupoid of invertible functors between C and $\mathcal{D}$. If C is a point then $Fun^0(C, \mathcal{D})$ is the maximal groupoid in $\mathcal{D}$. If $\mathcal{D}$ is to be in the truncation then this groupoid must be a discrete space, i.e. have only identity maps.

Axioms for C to be a presentable theory of (∞, n)-categories

First Axiom: $\tau_{\leq 0} \simeq Gaunt_n$ and this strongly generates C, i.e. for all $X \in C$:

$$\text{hocolim}_{D \in Gaunt_n, D \to X} \xrightarrow{\simeq} X$$

Second Axiom: cells detect equivalences, i.e. for any $X \in C$ then $f : X \simeq Y$ iff $C(C_i, X) \simeq C(C_i, Y)$ for all cells C_i. Also, cells have internal homs. Equivalently, the functor $X \times_{C_i} - : C/C_i \to C$ preserves homotopy colimits.

Third Axiom: In C a certain finite list of colimit equations is satisfied. These are obvious if you read up on Θ_n-spaces.

Fourth Axiom: C is universal with respect to the list above, i.e. if $\mathcal{D}$ satisfies the axiom then there's a localization $L : C \to \mathcal{D}$.

Note that all the axioms are satisfied by all the known examples discussed in this talk, as well as by Θ_n-spaces (so in particular Θ_n-spaces and CSS$_n$ are equivalent).

4. Q & A

As soon as the first condition is satisfied you can classify models with a certain property via localizations which preserve gaunts. For example, you can axiomatize stable (∞, n)-categories.

There are also axioms for (n, n)-categories. It's all the same axioms as above, but with one more colimit equation in the third axiom.