Mike Hill: Derived Equivariant Algebraic Geometry (or, e-DAG)

This talk will be all questions and no answers. We'll explain our confusion, we'll explain why you're confused and why you should care (a form of megalomania), and then we'll see why everything we thought was wrong. Everything. We'll see that the Zariski definition fails, we have a candidate definition for etale, but we'll see that it's tough to get a handle on.

We begin with an observation: suppose $\mathcal{O}$ is a sheaf of commutative rings on a stack $\mathcal{M}$, and suppose we have a cover $X \to \mathcal{M}$ on which we can evaluate $\mathcal{O}$. There should be a group G of "deck transformations" of $X \to \mathcal{M}$; we'll only think about finite groups, or maybe even only finite cyclic groups. But so, by naturality, we get an action of G on $\mathcal{O}(X \to \mathcal{M})$. Surprisingly, this is often a genuine equivariant commutative ring spectrum. (A priori it's just a naive equivariant ring spectrum, i.e. a G-object in commutative ring spectra. But rather, this gives us transfers and norms, and the homotopy groups obtain a much-enriched structure.) We motivate this with two examples. Actually, these are all the examples we've got.

Example 5. Take $\mathcal{M} = \mathcal{M}_{ell}$, the moduli stack of elliptic curves, and take $\mathcal{O} = \mathcal{O}^{Der}$ to be the Goerss–Hopkins–Miller sheaf giving tmf. Now, let's look at the moduli problem with level structure, say $\mathcal{M}_1(3)_{(2)} \to \mathcal{M}_0(3)_{(2)}$ (we're all 2-local here; we'll omit that hereafter). Recall that the source parametrizes an elliptic curve with a point of exact order 3, whereas the target parametrizes an elliptic curve with a subgroup of order 3. So this is a Galois cover, with a C_2-action by the sign action (i.e. by taking the chosen point to its negative).

Now, by definition, $\mathcal{O}^{Der}(\mathcal{M}_0(3)) = TMF_0(3)$ and $\mathcal{O}^{Der}(\mathcal{M}_1(3)) = TMF_1(3)$, and we have $TMF_0(3) \simeq TMF_1(3)^{hC_2}$ (in ring spectra). So, this could be naively pushed into the equivariant setting. But on the other hand, we could also construct $TMF_1(3)$ as a genuine ring in the first place. Note that $\pi_* TMF_1(3) = \mathbb{Z}_{(2)}[a_1, a_3][\Delta^{-1}]$, with $|a_i| = 2i$. The key observation here is that these generators have *equivariant refinements* $\overline{a}_i : S^{i \cdot \rho_{C_2}} \to TMF_1(3)$. From this we can rebuild the homotopy fixedpoints computation e.g. by the slice spectral sequence.

Moreover, if we compactify $\mathcal{M}_0(3)$, then the induced compactification of $\mathcal{M}_1(3)$ will no longer be a double-cover: it will be branched precisely at the compactification locus. We can still build commutative ring spectra out of these, but we no longer have a Galois cover here. If we use these equivariant lifts $\overline{a}_i$, we can hope to rebuild the story internally to a theory of equivariant (derived) algebraic geometry.

Example 6. The only other example we've got is the same story with the cover $\mathcal{M}_1(5) \to \mathcal{M}_0(5)$, which is now a C_4-cover. But now, we have two generators, and the generator of C_4 switches these. But that makes it impossible to invert just one. So if we tried to construct even the coarse moduli space $\mathbb{P}^1$, we need to allow the group to act on the various rings, and on the point in the prime spectra, in a way which may not be reflected down in algebra itself.

We mention that the first computations involving TMF with level structures were done by Mahowald and Rezk, there's also cool work by Behrens and Ormsby, and a lot of this came out of conversations with Meier and Stojanoska.

Now, let's talk about Spec. Let's take $G = C_2$, and let's consider $\underline{A}(C_2/C_2) = \mathbb{Z}[t]/(t^2 - 2t)$. What's Spec of this? Well, let's walk through it. If we invert 2, we just get $\operatorname{Spec}(\mathbb{Z}[1/2]) \sqcup \operatorname{Spec}(\mathbb{Z}[1/2])$. This is just from the Chinese remainder theorem. On the other hand, if we reduce mod 2, we get a fat point. So all in all, this is two copies of $\operatorname{Spec}(\mathbb{Z})$, which are tangent at their respective copies of (2).

Note that this doesn't have a generic point: (0) isn't prime, since e.g. t and $t-2$ are zerodivisors.

One reason we might care about this is tom Dieck's computation that $\pi_0((S^0)^{C_2}) = \underline{A}(C_2/C_2)$. Of course, we should think of the left side as $[S^0, S^0]^G$. If we reconsider this source as $S^0 = S^0 \wedge S^0 = S^0 \wedge (C_2/C_2)_+$, we see that this actually collects into a contravariant functor $\mathtt{Set}^G \to \mathtt{Ab}$ given by $T \mapsto [T_+ \wedge S^0, S^0]^G$. What tom Dieck actually showed is that this is $\underline{A}(T)$ (equivalently, the group-completion of the category of finite G-sets over T) – and of course, all of this collects into a Mackey functor: in this context, equivariant Spanier–Whitehead duality (or "Atiyah duality") tells us that we have $[T_+ \wedge S^0, S^0]^G \cong [S^0, T_+ \wedge S^0]^G$, so we get a covariant functor too.

Now, we should think of Mackey functors (like $\underline{A}$) as being like the abelian groups in our setting. These have a "tensor product", which is really just the Day convolution with respect to $(\mathtt{Set}^G, \times)$ and $(\mathtt{Ab}, \otimes)$. Thus, we Dayly convolve (i.e., convolve in the sense of Day) the cartesian product of finite G-sets with the tensor product of abelian groups.

Here's a way we can get at some Mackey functors. Define $\underline{A}_T = \underline{A}(T \times -)$. These turn out to be *projective* objects, and these are already "enough": using judicious choices for T, we can resolve any Mackey functor using the $\underline{A}_T$. Moreover, it's totally formal that $\underline{A}_T \boxtimes \underline{A}_S \cong \underline{A}_{T\times S}$. So it's actually not so hard to compute derived tensor products of Mackey functors.

Definition 5. A *commutative Green functor* is a commutative monoid for the $\boxtimes$ monoidal structure.

Let's attempt to specify a Green functor $\underline{R}$. We should have that $\underline{R}(G/H)$ is a commutative ring, and for $H \leq K$, the induced map $G/H \to G/K$ gives us a *restriction* map $\mathrm{res}_H^K : \underline{R}(G/K) \to \underline{R}(G/H)$ and a *transfer* map $\mathrm{tr}_H^K : \underline{R}(G/H) \to \underline{R}(G/K)$. We need the restriction map to be a ring map; this endows $\underline{R}(G/H)$ with the structure of an $\underline{R}(G/K)$-bimodule, and then the transfer map should be a map of bimodules. All of this can also go under the name of *Frobenius reciprocity*: $a \cdot \mathrm{tr}_H^K(b) = \mathrm{tr}_H^K(\mathrm{res}_H^K a \cdot b)$. (Of course once we're commutative we don't need to worry about *bi*modules, but this now transfers to the non-commutative case too.)

Now, all the homotopy groups of any G-spectrum are Mackey functors, and moreover for a commutative ring spectrum then $\underline{\pi}_0$ is a commutative Green functor (and the other $\underline{\pi}_i$ are modules for it). Our favorite ring spectrum is just S^0, whose 0^{th} homotopy is the Burnside ring. This is ring-valued, so we can unpack it as an example.

Example 7. We have $\underline{A}(C_2/e) = \mathbb{Z}$, the restriction $\underline{A}(C_2/C_2) \to \underline{A}(C_2/e)$ is given by $t \mapsto 2$, and the transfer is given by $1 \mapsto t$. Then, the relation $t^2 - 2t = 0$ is precisely Frobenius reciprocity!

Brun first showed that for an arbitrary commutative ring G-spectrum X, we have that $\underline{\pi}_0 X$ is a Tambara functor. Tambara called these "TNR functors", for "transfer", "restriction", and "*norms*". These last are the difference between Green functors and Tambara functors. These are *multiplicative, non-additive* maps $N_H^K : \underline{R}(G/H) \to \underline{R}(G/K)$. First of all, the norm and restriction interact as if we were making a multiplicative version of the Mackey functoriality condition. But then, there are also formulas that tell us how the norm interacts with sums and with transfers. (We mention that Strickland has a beautiful and detailed memoir, in which he describes this from a categorical perspective. We're working more explicitly, which helps for computations but makes it harder to prove theorems.) The important one for us is given by

$$N_H^K(a+b) \equiv N(a) + N(b) \text{ mod transfers.}$$

One should think of the norm as $N(a) = \prod_{g\in G} g \cdot a$, and then this yields

$$N(a+b) = \prod_{g\in G}(a+b) = \prod_{g\in G}(ga+gb);$$

in expanding this out, we get a bunch of terms with only a's or b's, and these are the norms of a and b themselves, and then the mixed terms are all transfer terms. In fact, we'll need to make this even more concrete in our toy example:

$$N_e^{C_2}(a+b) = N_e^{C_2}(a) + N_e^{C_2}(b) + \mathrm{tr}_e^{C_2}((ga)\cdot b).$$

Let's see what this buys us in the Burnside ring $\underline{A}$. There, for a prime p, we have that

$$N_e^{C_2}(p) = p + \frac{p^2-p}{2} \cdot t.$$

(This follows by induction.) This behaves weirdly precisely at $p = 2$: $N_e^{C_2}(2) = 2 + t$. At all odd primes, this would still be divisible by p.

Now we can finally make a definition of Spec: this is of course the set $\{\mathfrak{p} : \mathfrak{p} \text{ is prime}\}$. But what is an ideal? Now, this is something that we can quotient by and still get a Tambara functor. Let us call an ideal $\underline{I}$ in a Tambara functor a *Tambara ideal* if it's a Green ideal and $N_H^K \underline{I}(G/H) \subset \underline{I}(G/K)$. For example, inside of $\underline{A}$, taking (p) at both slots is a Tambara ideal iff $p > 2$.

Now, Nakaoka defines a Tambara ideal $\mathfrak{p}$ to be *prime* if, whenever $(a)_{Tamb}\dot{(b)}_{Tamb} \subset \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, where the notation denotes the *Tambara* ideal generated by the element: these are no longer principal Green ideals! For instance, $(2)_{Tamb} = (2, t)$. Then, Nakaoka proves that for *any* group G, the Burnside Tambara functor is an *integral domain*: the ideal (0) is prime in $\underline{A}$.

This gives us a way to define Spec, copying all the normal definitions; we denote it by Spec^T. For instance, at $G = C_2$ we have (in the same pattern as before): 0 in $\underline{A}(C_2/C_2)$ and $\underline{I} \subset \underline{A}(C_2/e)$; only $(2)_{Tamb}$ at the even prime; and then for all odd primes p, (p) and $(p) + \underline{I}$.

Here's something really annoying: the complement of a prime Tambara ideal is not multiplicatively closed! This prevents us from defining a good sheaf of rings covering this space.

We note that $\underline{A} \otimes \mathbb{Q}$ also has an interesting Spec^T. Even for a cyclic group, we get an interesting topology, a sort of "solar system". The rationalized G-symmetric monoidal category has only a *filtration*, but not a splitting (which only happens when we forget down to a plain symmetric monoidal category).

We don't have a splitting that preserves the norms. But another perspective is that we might think of the norms as giving *extra structure* on the category of G-spectra: it is not just symmetric monoidal, but G-symmetric monoidal. Doing this for the wedge and cartesian products gives the same thing; this is equivariant stability. But doing it for the smash product is what gives $\underline{\pi}_0$ of a commutative ring a Tambara functor structure. But this is also what fails, and makes the category not split even in the rational case: if we forget back down to a symmetric monoidal category, then we're just doing Spec in Green functors!