Vesna Stojanoska: Duality, algebro-homotopically

We begin with R being a commutative ring.

Definition 6. An R-module K is called *dualizing* if:

1. for all $M \in \mathtt{Mod}_R$ finitely generated, $M \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_R(M, K), K)$; and
2. K has finite injective dimension over R.

The second condition means that in the derived setting, the first condition has good finiteness properties: if M is suitably connective, then $\mathrm{Hom}_R(M, K)$ is bounded above. (Note that for instance, in spectra, we just need to check the first condition on $M = S^0$.)

When R is an E_∞-ring, we add the following condition:

3. for all i, $\pi_i K$ is finitely generated over $\pi_0 R$.

However, we also need to be careful with condition 2 in the case that R is nonconnective (for instance, if it's periodic).

A question from the audience: In algebraic geometry, one talks about *the* dualizing complex. Here, is there no preferred choice? Answer: There is a preferred choice in the *relative* context.

Example 8. For $R = k$ a field, Hk is a dualizing Hk-module.

Example 9. For $R = S^0$, S^0 is not dualizing over itself: it doesn't have finite injective dimension. However, we happen to know another spectrum that *is* dualizing: the Anderson dualizing spectrum $I_{\mathbb{Z}}$ is a dualizing spectrum. Then, for X any spectrum, we can compute $\pi_* I_{\mathbb{Z}} X$ from $\pi_* X$ via a spectral sequence $\mathrm{Ext}^s_{\mathbb{Z}}(\pi_t X, \mathbb{Z}) \Rightarrow \pi_{-t-s}\mathcal{F}(X, I_{\mathbb{Z}}) = \pi_{-t-s} I_{\mathbb{Z}} X$. This comes from the fiber sequence $I_{\mathbb{Z}} \to H\mathbb{Q} \to I_{\mathbb{Q}/\mathbb{Z}}$ (where the last term is the Brown–Comenetz dualizing spectrum).

Let's return to the question of when R is nonconnective. We can try to fix the definition as follows.

1. Find what adjunctions result from a spectrum being dualizing, and then just take those as the definitions.
2. We can use a "local" approach (this was especially pursued by Dwyer–Greenlees–Iyengar).

Here are a few questions [ed.: actually just one].

1. **Q.:** Why do we care? **A1.:** Why not?? Dualities are always exciting! **A2.:** According to a remark of Dan Freed, Anderson duality actually appears in nature. **A3.:** As we'll see, one can use duality in computations of Hopkins's $K(n)$-local Picard groups.

Let's return to attempted definition fix 1. Suppose that we have a map $f : X \to Y$ (of schemes, stacks, ...). This gives us a "relative global sections" functor $f_* : \mathtt{Mod}_{\mathcal{O}_X} \to \mathtt{Mod}_{\mathcal{O}_Y}$. This always has a left adjoint f^*, and in special situations it also has a right adjoint $f^!$. In such cases (and additionally if f_* is faithful), we can build duality on X using duality on Y. More precisely, if K_Y is a dualizing $\mathcal{O}_Y$-module, then $f^! K_Y$ will be a dualizing $\mathcal{O}_X$-module.

Now, suppose we have some derived algebro-geometric object $X = (X, \mathcal{O}_X^{der})$. The terminal object is $\mathrm{Spec}\, S^0$, and since we have duality in that case, we can often pull this back to duality on X.

Example 10. Serre computed that the terminal map $f : \mathbb{P}^n \to \mathrm{Spec}\, \mathbb{Z}$ admits $f^!$, and moreover $f^! \underline{\mathbb{Z}} \cong \Lambda^n \Omega_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$ (the sheaf on $\mathbb{P}^n$ associated to $\Sigma^{-(n+1)}\mathbb{Z}[x_0, \ldots, x_n]$, considering $\mathbb{P}^n := \mathrm{Proj}\ \mathbb{Z}[x_0, \ldots, x_n]$).

Here's an upside: dualizing modules are *uniformly* described, since Kahler differentials have nothing to do with duality a priori.

But here's a downside: if you add stacky points to something with duality, you'll ruin the duality. This is because adding stackiness introduces group cohomology to computations, and (even in the finite group case) this is infinite-dimensional, which makes us lose track of dualities.

We should see the previous example as an *advertisement* for derived algebraic geometry. The downside becomes an upside: if we add stacky points that come from $\mathcal{M}_{fg}$, this fixes the problem. (This is just a heuristic, but it's strongly supported especially by recent work of Mathew–Meier.) We will explain this more through a sequence of examples.

Example 11. Let's take the simplest possible nontrivial group: $BC_2 \xrightarrow{f_0} \operatorname{Spec}\mathbb{Z}$. There's a homotopical refinement of this: BC_2 admits an even-periodic spectrification to KU, which we'll denote by $(X, \mathcal{O}_X) = (BC_2, KU)$, and we replace $\operatorname{Spec}\mathbb{Z}$ by $\operatorname{Spec} S^0$ to get $(BC_2, KU) \xrightarrow{f} \operatorname{Spec} S^0$. Now, $(f_0)_*$ is way more poorly behaved than f_*. Now, note that $\mathcal{O}_X$-modules are like KU-modules with a compatible C_2-action, i.e. KO-modules. By joint work with Drew Heard, the right adjoint $f^!$ of f_* is such that $f^!(A) = \mathcal{F}(KU, A) \simeq \mathcal{F}(I_{\mathbb{Z}}A, \Sigma^4 KU)$ (an equivalence of C_2-equivariant spectra). The key to the prove is the norm equivalence $KU_{hC_2} \xrightarrow{\sim} KU^{hC_2}$. (The obstructions to this equivalence vanish, since they live in the gap in $\pi_* KO$.)

Example 12. Consider $(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{Der}) \xrightarrow{f} \operatorname{Spec} S^0$. By Mathew–Meier, f_* has a right adjoint (since the source actually behaves affinely, at least as far as quasicoherent sheaves are concerned). In our thesis, we showed that $f^! I_{\mathbb{Z}} \simeq \Sigma^{21}\mathcal{O}^{der}$. To get an analog of the theorem above, though, we need to build in e-DAG. This will be harder than what Mike Hill was talking about, because we'll want to work with $\overline{M}(n) \to \overline{M}_{ell}$, which has deck group $GL_2(\mathbb{Z}/n)$ (which is no longer cyclic).

We now say a bit about computations of Pic_n, the group of equivalence classes of $K(n)$-local smash-invertible spectra. This always has an *algebraic* component Pic_n^{alg} (in which the elements are entirely detected by their completed E_n-homology), and for $p \gg n$ this is everything but otherwise there is an extension by an *exotic* component κ_n.

Now, Gross–Hopkins duality tells us that $L_{K(n)} I_{\mathbb{Z}} \in Pic_n$. We can ask: Is this an exotic element? As can be seen in Strickland's note on Gross–Hopkins duality, writing $I_n = L_{K(n)} I_{\mathbb{Q}/\mathbb{Z}} \simeq L_{K(n)}(\Sigma I_{\mathbb{Z}})$, we have that $E_n \wedge I_n \simeq \Sigma^{n^2-n} E_n\langle \det \rangle$, and this is algebraic.

Example 13. Our previous example with KU and KO shows that we have $\mathbb{Z}/2 \subset \kappa_1$ at $p = 2$. This is because if I_1 were algebraic, then (computing $K(1)$-locally) we'd have

$$\Sigma^6 KO \simeq \Sigma \mathcal{F}(KO, I_1) \simeq KO \wedge I_1 \simeq (KU \wedge I_1)^{hC_2} \xrightarrow{\sim ?} (\Sigma^2 KU)^{hC_2} \simeq \Sigma^2 KO.$$

That is, KO would be 4-periodic, which is not true (even $K(1)$-locally).

Likewise, the fact that tmf is Anderson-self-dual up to a 21-fold suspension, that example implies that there is $\mathbb{Z}/3 \subset \kappa_2$ at $p = 3$ (and we're about 80% sure that we also have $\mathbb{Z}/8 \subset \kappa_2$ at $p = 2$).

Of course, one big goal from here is to get more of these sorts of examples at higher heights. A more challenging goal is to actually get *all* of the κ_n's by some general construction.

In the last few minutes, we'll say a few words about doing duality "locally" (in the sense of Dwyer–Greenlees–Iyengar). This is based on the theory of *residual complexes*. This is well-known in classical algebraic geometry, and this is what DGI have made sense of in topology, but only locally thus far.

Example 14. Consider the diagram

$$\operatorname{Spec}\mathbb{F}_p \xrightarrow{x_p} \operatorname{Spec}\mathbb{Z} \xleftarrow{\xi} \operatorname{Spec}\mathbb{Q}.$$

Now, $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, and the injective hull of $\mathbb{F}_p$ is $\mathbb{Z}/p^\infty$. So we can take the injective resolution

$$\mathbb{Q} \to \bigoplus_p \mathbb{Z}/p^\infty \cong \mathbb{Q}/\mathbb{Z}$$

as a dualizing complex.

Example 15. The previous example works also for curves. Say $X \to \operatorname{Spec} k$ is a proper smooth curve. Call ξ its generic point, and there we have the dualizing module $\Omega^1_{X,\xi}$. Similarly, we can construct

$$\Omega^1_{X,\xi} \to \bigoplus_{x \in X \setminus \{\xi\}} \Omega^1_{X,\xi}/\Omega^1_{X,x}.$$

This is again a dualizing complex.

One thing that's a bit unclear to us about this story is that in homotopy theory we have lots of primes (the Morava K-theories), and this suggests the following question: What are residual complexes in homotopy theory?