Charles Rezk: Calculations in multiplicative stable homotopy theory at height 2

We'll give a broad overview, and then describe a calculation that you can get your hands on. There won't be much reimagining in this talk, and we apologize for that.

Definition 7. Recall that a *formal group* over a ring A is a formal scheme G with $\mathcal{O}_G \cong A[[x]]$, equipped with a group structure (corresponding to $\mathcal{O}_G \to \mathcal{O}_G \hat{\otimes}_A \mathcal{O}_G$).

An *isogeny* is a map $f : G \to G'$ of formal groups such that the induced map $f^* : \mathcal{O}_{G'} \to \mathcal{O}_G$ is finite and locally free.

In the case that $f : G \to G'$ is an isogeny, we can define $K = \ker(f)$: this has $\mathcal{O}_K \simeq \mathcal{O}_G \otimes_{\mathcal{O}_{G'}} A$, and this will be a finite commutative group scheme over A. We define the *degree* of f to be $\deg(f) = \mathrm{rk}_A \mathcal{O}_K$.

Example 16. Here's a non-example. Consider $[p] : \hat{\mathbb{G}}_m \to \hat{\mathbb{G}}_m$ over $\mathbb{Z}$. Then $[p]^*(x) = px + \cdots + x^p$. This is *not* an isogeny. However, applying $- \otimes \mathbb{Q}$ turns this into an isogeny of degree 1, and applying $\otimes \mathbb{Z}_p$ turns this into an isogeny of degree p.

The moral is that we want our maps to be isogenies, and for this to happen we often need to restrict.

Example 17. Given $\mathbb{F}_p \subset A$ and G/A a formal group, we have $F^r : G \to (\varphi^r)^* G$, defined locally by $x \mapsto x^{p^r}$. This is even better when $A = k$ is a *field*. Then a degree-p^r isogeny $f : G \to G'$ factors uniquely as

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "F^r"'] & G' \\ (\varphi^r)^* G. \arrow[ur, dashed, "\sim"] & \end{tikzcd}$$

To go further, we'd like to talk about *deformations*.

Definition 8. Fix G_0/k, for k a perfect field of characteristic p. Then $[p]$ is an isogeny of degree p^n (here n is called the *height* of G_0.) Then, we define a *deformation structure* to be, given a formal group G over a complete local ring A, an element of the set $\mathcal{D}(G/A) = \{(i, \alpha) : i : k \to A/\mathfrak{m}, \alpha : i^* G_0 \xrightarrow{\sim} G_{A/\mathfrak{m}}\}$.

Now, given an isogeny $f : G \to G'$ over A, we can pullback deformation structures: we get $f_* : \mathcal{D}(G/A) \to \mathcal{D}(G'/A)$. This follows from the "unique factorization through an iterated Frobenius" result above: we have

$$\begin{tikzcd} i^* G_0 \arrow[r, dashed, "F'"] \arrow[d, "\alpha"', "\sim"] & (\varphi^r)^* i^* G_0 \arrow[d, "\exists!\ \alpha'"] \\ G_{A/\mathfrak{m}} \arrow[r] & G'_{A/\mathfrak{m}}. \end{tikzcd}$$

So, $f_*(i, \alpha) = (i \circ \varphi^r, \alpha')$.

Exercise 1. Check that for $\mathbb{F}_p \subset A$, $F_* = \varphi^* : \mathcal{D}(G/A) \to \mathcal{D}(\varphi^* G/A)$.

We would like to collect these deformations all together.

Definition 9. Fix G_0. Given a complete local ring A, we have a *category* $\mathtt{Def}_{G_0}(A)$, whose objects are pairs $(G/A,\ (i, \alpha) \in \mathcal{D}(G/A))$, and whose morphisms are isogenies $f : G \to G'$ that are compatible with the choice of deformation structure. This is functorial in local homomorphisms: given $g : A \to A'$, we get $g^* : \mathtt{Def}(A) \to \mathtt{Def}(A')$. (This is covariant, but we think scheme-theoretically.)

Altogether, this is a functor $\mathtt{Def} : \mathtt{CpltLocRing} \to \mathtt{Cat}$; we can think of this as a category-valued presheaf on a certain category of formal schemes. This is not a (pre-)stack, because these need to be valued in groupoids; thus, instead we call them (pre-)*piles*. (The terminology can be blamed on Matt Ando.)

Let us now talk about quasicoherent sheaves. The category $\mathtt{QCoh}(\mathtt{Def})$ is $M = \{M_A, M_g\}$: for any complete local ring A we want $M_A : \mathtt{Def}(A)^{op} \to \mathtt{Mod}_A$, and for a local homomorphism $g : A \to A'$ we want a natural equivalence $M_g : A' \otimes_A M_A \xrightarrow{\sim} M_{A'} \circ g^*$. (These data should satisfy some coherence conditions, of course.)

Example 18. Define $\omega \in \mathtt{QCoh}(\mathtt{Def})$ by $\omega_A(G/A)$, the rank-1 A-module of invariant 1-forms. Associated to an isogeny $f : G \to G'$, we simply pull back 1-forms from G' to G.

Example 19. Here is a stupider example, $\deg \in \mathtt{QCoh}(\mathtt{Def})$. Define $\deg_A(G/A) = A$, and given $f : G \to G'$ we define f^* to be multiplication by $\deg(f) \in \mathbb{Z}$.

Let us digress for a minute to talk about elliptic curves. We could replace complete local rings by schemes, we could replace the formal groups with elliptic curves, and we could keep using isogenies. This gives us a pile $\mathtt{Ell}^{isog}$. (One could also restrict to p^{th}-power isogenies, giving $\mathtt{Ell}^{p-isog}$.) These have notions of quasicoherent sheaves, and once again we can write down examples.

For instance, associated to an elliptic curves C/S we have $H^k_{dR}(C/S)$ (the algebraic de Rham cohomology sheaf over S). This yields a hypercohomology spectral sequence (a/k/a the "algebraic Hodge-to-de Rham spectral sequence") for $H^k_{dR} \in \mathtt{QCoh}(\mathtt{Ell}^{isog})$. This actually degenerates, and we have $H^0_{dR}(C/S) \simeq \mathcal{O}_S$, and we have an exact sequence

$$0 \to H^0(\Omega^1_{C/S}) = \omega \to H^1_{dR}(C/S) \to H^1(\mathcal{O}_{C/S}) = \omega^{-1} \otimes \deg \to 0.$$

We also can compute that $H^2_{dR}(C/S) \simeq \deg$ (coming from the fundamental class on the elliptic curve).

In particular, we note that the above short exact sequence gives us a *Hodge class*, an element of $\mathrm{Ext}^1_{\mathtt{Ell}}(\omega^{-1} \otimes \deg, \omega)$. We also get e.g. an element of $\mathrm{Ext}^1_{\mathtt{Ell}^{p-isog}_{\mathbb{C}}}(\omega^{-1} \otimes \deg, \omega)$; this comes with an injection from $MF_{wt=2}(\Gamma_0(p))^{w=-1}$, where w is the Atkin–Lehner involution and we're picking out the (-1)-eigenspace. This takes $E_{2,p}(q)$ to the Hodge class. (Here, $E_{2,p}(q) = E_2(q) - pE_2(q^p)$, where $E_2(q) = (-1/12) + \sum_{d|n} dq^n$.)

We remark that this should have something to do with viewing elliptic cohomology as an "ultracommutative" global-equivariant ring. (This will require both e-DAG and plethories.)

Now, fix G_0/k. We defined a set $\mathcal{D}(C/A)$; this carries a *free* action of $\mathrm{Aut}(G/A)$, and so in the category $\mathtt{Def}(A)$, there is at most one isomorphism. Thus the quotient $\mathtt{Def}(A) \to \mathtt{Def}(A)/\sim$ is actually an equivalence of categories.

Now, $\mathtt{Def}$ has a sort of grading, because we've got the notion of the *degree* of a morphism. We define $\mathtt{Def}^r(A)$ to be the subcategory whose morphisms all have degree r. Then, the objects of $\mathtt{Def}(A)/\sim$ are in 1-to-1 correspondence with $\{(G \supset K), G \in \mathtt{Def}^0/\sim\}$ (for K a subgroup of degree p^r), given by taking the kernel.

Theorem 4 (Lubin–Tate, Strickland)**.** *$\mathtt{Def}$ is representable: there exist complete local rings A_r that represent $\mathtt{Def}^r(A)/\sim$.*

(The case $r = 0$ is due to Lubin–Tate, with $A_0 \cong W_pk[[u_1, \ldots, u_{n-1}]]$; Strickland extended to the general case. Namely, A_r is the object that classifies certain subgroups of formal groups.)

Theorem 5 (Morava, Goerss–Hopkins–Miller, Strickland)**.** *There is a unique commutative S-algebra $E = E_{G_0/k}$ such that $E^*B\Sigma_{p^r}/I \simeq A_r[u^{\pm}]$ for $|u| = 2$, where I is the ideal generated by transfers from proper subgroups $\Sigma_i \times \Sigma_{p^r-i} \subsetneq \Sigma_{p^r}$.*

(Again, for $r > 0$ this is due to Strickland.) This all yields a graded category object $\{A_r\}$ in $\mathtt{CpltLocRing}^{op}$. Now, there is a correspondence between $\mathtt{QCoh}(\mathtt{Def})$ and a certain category of comodules, which takes $\underline{M} \in \mathtt{QCoh}(\mathtt{Def})$ to $M \in \mathtt{Mod}_{A_0}$ and the maps $M \xrightarrow{\psi_r} \prod_{r \geq 0} A_r \otimes_{A_0} M$.

This is all what we need to talk about *power operations*. Let R be a commutative, $K(n)$-local E-algebra. Then (writing $m = p^r$) we have the composite ring homomorphism

$$\psi_r : \pi_0 R \xrightarrow{P_m} \pi_0 R^{B\Sigma_m^+} \simeq \pi_0 R \otimes_{E_0} E^0 B\Sigma_m \to \pi_0 R \otimes_{A_0} A_r.$$

Thus, we can consider $\pi_0 R \in \mathtt{QCoh}(\mathtt{Def})$.

Now, there exists a category $\mathcal{A} = \mathcal{A}_{G_0}$. In the language of Clark Barwick's talk, this is the category of algebras for a certain E_*-plethory. Roughly speaking, this comes from $\pi_* : \mathtt{CAlg}(E)_{K(n)} \to \mathtt{Mod}(E_*)$, and we can lift this functor to $\pi_* : \mathtt{CAlg}(E)_{K(n)} \to \mathcal{A}$. So, this is essentially a category of quasicoherent sheaves of (graded) *rings* on $\mathtt{Def}$, although there are some further conditions (including a "Frobenius congruence"). More precisely, $\mathcal{A}$ is an equivalence on *p-torsion-free* objects. In fact, $\mathcal{A}$ has free objects, and these happen to be p-torsion-free. So we can actually reconstruct $\mathcal{A}$ from an understanding of $\texttt{QCoh}(\texttt{Def}, \texttt{Ring}^*)_{Frb}$.

The idea of the Frb is as follows. Given $R \in \texttt{QCoh}(\texttt{Def}, \texttt{Ring})$ (with $\mathbb{F}_p \subset A$) satisfies the *Frobenius congruence* if, when we evaluate R to get $R_A(G,(i,\alpha))$, the degree-p Frobenius isogeny gives us

$$R_A(\varphi^*G, F_*(i,\alpha)) \xrightarrow{F} R_A(G,(i,\alpha)),$$

and the source of this map is

$$R_A(\varphi^*G, F_*(i,\alpha)) \simeq R_A(\varphi^*G, \varphi^*(i,\alpha)) \simeq A^{\varphi} \otimes_A R_A(G,(i,\alpha)).$$

This last object has the usual relative Frobenius down to $R_A(G,(i,\alpha))$, and R satisfies the *Frobenius congruence* if this diagram commutes. (This essentially amounts to checking something mod p, which is why we call it a "congruence".)

Example 20. Let $G_0 = \hat{\mathbb{G}}_m/\mathbb{F}_p$. This has height 1. We have $A_r = \mathbb{Z}_p$, and $\texttt{QCoh}(\texttt{Def})$ is the category of $\mathbb{Z}_p$-modules M with an action of the Adams operations. Then, $\texttt{QCoh}(\texttt{Def}, \texttt{Ring})_{Frb}$ consists of rings R with an Adams operation ψ such that $\psi(x) \equiv x^p \bmod p$, and to keep track of this we simply add a function θ. These are called *θ-rings*, and indeed, $\mathcal{A}$ is the category of *graded θ^p-rings*.

Remark 1. The data of the graded category object $\{A_r\}$ (along with its extra structure) is in some sense *quadratic*: it is completely determined by A_1, the source and target maps $s,t : A_0 \to A_1$, and a composition map $A_2 \to A_1 \otimes_{A_0} A_1$. Everything else is (co)generated by this data.

Remark 2. The category $\texttt{QCoh}(\texttt{Def})$ has finite homological dimension: in fact, it is $2n$. Furthermore, there is a "Koszul complex" way of building explicit resolutions. (For instance, at height 2 there's a 3-term complex one can use to compute Ext, as long as the underlying modules are projective over A_0.) It turns out that one can build the exactly same complex for elliptic curves.

Remark 3. Somewhat annoyingly, at heights at least 3, we only know proofs for both of the previous remarks that go through topology (instead of staying in algebra).

We end with an example.

Example 21. Given R and F two augmented $K(n)$-local commutative E-algebras, there's a spectral sequence

$$E_2^{s,t} \Rightarrow \pi_{t-s}\mathrm{map}_{\texttt{CAlg}(E)_{/E}}(R,F).$$

If π_*R is even-concentrated and *smooth*, this E_2-term is given by

$$E_2^{0,0} = \mathrm{Hom}_{\mathcal{A}_{/\pi_*E}}(\pi_*R, \pi_*F),$$

and for $s > 0$,

$$E_2^{s,t} = \mathrm{Ext}^s_{\texttt{QCoh(Def)}}(\omega^{-1} \otimes \hat{Q}(\pi_*R), \omega^{(t-2)/2} \otimes \pi_*\overline{F}).$$

Here's an example result; this is a special case of a conjecture of Hopkins and Lurie (which may actually now be a theorem).

Proposition 1. *Let $G_0/\overline{F}_p$ have height 2. Then,*

$$\mathrm{map}_{\texttt{CAlg}(S^0)}(\Sigma^\infty_+, E) \simeq \overline{\mathbb{F}}_p^\times \times K(\mathbb{Z}_p, 3).$$

In this case, the spectral sequence degenerates. There's $E_2^{0,0} = \overline{\mathbb{F}}_p^\times$, and the $E_2^{s,t}$ vanishes otherwise except for $E_2^{1,4} = \mathrm{Ext}_{\texttt{QCoh(Def)}}(\omega^{-1} \otimes \deg, \omega) \cong \mathbb{Z}_p$. This is actually generated by the Hodge class under the image of a natural homomorphism from $\mathrm{Ext}_{\texttt{QCoh}(\texttt{Ell}^{isog})}$ (of the same objects); this is visible in the work of Nick Katz.

One can also compute that $\mathrm{map}_{\texttt{CAlg}(S^0)}(\Sigma^\infty_+\mathbb{Z}, TMF_p) \simeq \mathbb{Z}_p^{c_p}$, where c_p counts supersingular elliptic curves in a certain way: $c_p = \dim MF_{wt=2}(\Gamma_0(p))^{w=-1}$.

MSRI TALK, APRIL 10, 2014

CHARLES REZK

1. Isogenies

Formal group G/A:

$$\mathcal{O}_G \approx A[\![x]\!], \qquad x \mapsto F(x_1, x_2)\colon \mathcal{O}_G \to \mathcal{O}_G \widehat{\otimes}_A \mathcal{O}_G.$$

Isogeny: $f\colon G \to G'$ such that $f^*\colon \mathcal{O}_{G'} \to \mathcal{O}_G$ is finite locally free.

$\Longrightarrow K = \operatorname{Ker}(f)$, $\mathcal{O}_K = \mathcal{O}_G \otimes_{\mathcal{O}_{G'}} A$ is finite locally free over A. $\deg(f) = \operatorname{rank}_A \mathcal{O}_K$.

Example? $\widehat{\mathbb{G}}_m/\mathbb{Z}$.

$[p]\colon \widehat{\mathbb{G}}_m \to \widehat{\mathbb{G}}_m$, $[p]^*(x) = px + \cdots + x^p$.

Not an isogeny over $\mathbb{Z}$. Over $\mathbb{Q}$, isogeny of degree 1 (isomorphism). Over $\mathbb{Z}_p$, isogeny of degree p.

Frobenius isogeny. $\mathbb{F}_p \subseteq A$, any G/A,

$$F^r\colon G \to (\phi^r)^* G, \qquad x \mapsto x^{p^r},$$

degree p^r. ($\phi\colon A \to A$, $\phi(a) = a^p$.)

Over field $k \supseteq \mathbb{F}_p$. Unique factorization ($\deg f = p^r$).

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "F^r"'] & & G' \\ & (\phi^r)^* G \arrow[ur, "g"', "\sim"] & \end{tikzcd}$$

2. Deformations

Fix G_0/k: k perfect char p, $[p]_{G_0}$ isogeny of degree p^n. (Height n formal group.)

Deformation structures. Given G/A, $A =$ complete local ring.

$$\mathcal{D}(G/A) = \{\, (i, \alpha) \mid i\colon k \to A/\mathfrak{m},\ \ \alpha\colon i^* G_0 \xrightarrow{\sim} G_{A/\mathfrak{m}} \,\}.$$

Isogeny $f\colon G \to G'$ over $A \Longrightarrow f_*\colon \mathcal{D}(G/A) \to \mathcal{D}(G'/A)$:

$$\begin{tikzcd} i^* G_0 \arrow[r, "F^r"] \arrow[d, "\alpha"', "\sim"] & (\phi^r)^* i^* G_0 \arrow[d, dashed, "\exists!\, \alpha'", "\sim"'] \\ G_{A/\mathfrak{m}} \arrow[r, "f_{A/\mathfrak{m}}"'] & G'_{A/\mathfrak{m}} \end{tikzcd}$$

$f_*((i, \alpha)) = (i\phi^r, \alpha')$.

Exercise. For $\mathbb{F}_p \subseteq A$:

$$F_* = \phi^*\colon \mathcal{D}(G/A) \to \mathcal{D}(\phi^* G/A).$$

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Pile of deformation structures. $\mathrm{Def} = \mathrm{Def}_{G_0}$. A complete local ring $\Longrightarrow$

$$\mathrm{Def}(A) := \begin{cases} \textbf{obj}: & (G/A, \quad (i,\alpha) \in \mathcal{D}(G/A)), \\ \textbf{mor}: & f\colon G \to G', \quad f_*(i,\alpha) = (i',\alpha'). \end{cases}$$

Local homomorphism $g\colon A \to A' \Longrightarrow g^*\colon \mathrm{Def}(A) \to \mathrm{Def}(A')$.

Def: a presheaf of categories on $\{\text{cpt loc rings}\}^{\mathrm{op}}$. "Pile".

Quasi-coherent sheaves on Def**.** Objects of QCoh(Def) are $(\{M_A\}, \{M_g\})$:

$$A \quad \rightsquigarrow \quad M_A\colon \mathrm{Def}(A)^{\mathrm{op}} \to \mathrm{Mod}_A,$$
$$g\colon A \to A' \quad \rightsquigarrow \quad M_g\colon A' \otimes_A M_A \xrightarrow{\sim} M_{A'} \circ g^*.$$

Coherence, etc.

Example. $\omega \in \mathrm{QCoh}(\mathrm{Def})$.

$$\omega_A(G/A) := \{\text{invt 1-forms on } G\},$$

(rank 1 A-module). Forms pullback along homomorphisms.

Example. $\deg \in \mathrm{QCoh}(\mathrm{Def})$.

$$\deg_A(G/A) := A, \qquad f^* = \text{mult. by } \deg(p) \in \mathbb{Z}.$$

3. Digression: elliptic curves and isogenies

Formalism works more generally.

Pile of elliptic curves and isogenies. Ell.

Replace: complete local rings $\to$ schemes, formal groups and def str $\to$ ell curves, isog preserving def str $\to$ all isogenies.

Or just isogenies of pth power degree: Ell^p.

Example. Algebraic de Rham cohomology.

$$C/S \mapsto H^k_{\mathrm{dR}}(C/S), \qquad \text{coh sheaf over } S.$$

This is a functor, so gives object $H^k_{\mathrm{dR}} \in \mathrm{QCoh}(\mathrm{Ell})$.

Hypercohomology ss (algebraic "Hodge to de Rham").

$$H^0_{\mathrm{dR}}(C/S) \approx \mathcal{O}_S,$$
$$0 \to H^0(\Omega_{C/S}) \to H^1_{\mathrm{dR}}(C/S) \to H^1(\mathcal{O}_{C/S}) \to 0,$$

rewrite as

$$0 \to \omega \to H^1_{\mathrm{dR}}(C/S) \to \omega^{-1} \otimes \deg \to 0,$$
$$H^2_{\mathrm{dR}}(C/S) \approx \deg.$$

"Hodge class" in $\mathrm{Ext}^1_{\mathrm{Ell}}(\omega^{-1} \otimes \deg, \omega)$.

Remark. For $\mathrm{Ell}^p_{\mathbb{C}}$ (elliptic curves over $\mathbb{C}$ and p-isogenies), have inclusion

$$MF_{\text{weight}=2}(\Gamma_0(p))^{W=-1} \hookrightarrow \mathrm{Ext}^1_{\mathrm{Ell}^p_{\mathbb{C}}}(\omega^{-1} \otimes \deg, \omega).$$

$W =$ Atkin-Lehner involution.

Hodge class corresponds to $E_{2,p}(q) = E_2(q) - pE_2(q^p)$, where $E_2(q) = -\frac{1}{12} + \sum_{n,\, d|n} dq^n$.

Hodge class is non-trivial essentially "because" $E_2(q)$ is not a modular form. (Katz.)

Hope. We will note below that QCoh(Def) has something to do with Morava E-theory (as comm S-algebra).

Dream: QCoh(Ell) has similar relationship to elliptic cohomology, as a globally equivariant ultracommutative ring/scheme.

4. Def is representable; Morava E-theory

Fix G_0/k as before.
$\mathrm{Aut}(G/A)$ acts *freely* on deformation structures $\mathcal{D}(G/A)$.
$\Longrightarrow$ at most one iso between any two objects of $\mathrm{Def}(A)$ ($\mathrm{Def}(A)$ is "0-truncated" in Cat).
Can form $\mathrm{Def}(A)/\sim$: identify isomorphic objects. "Gaunt".
Let $\mathrm{Def}^r(A)/\sim :=$ set of morphisms of degree p^r. (If $r=0$, these are objects.)

$$\mathrm{Def}^r(A)/\sim \longleftrightarrow \{\, (G,K) \mid K \leq G \text{ subgroup of deg } p^r \,\}.$$

4.1. **Theorem** (Lubin-Tate, Strickland)**.** *There exist complete loc rings A_r, $r \geq 0$, so*

$$\mathrm{Hom}(A_r, B) \approx \mathrm{Def}^r(B)/\sim .$$

(Local homomorphisms.) Isomorphism $A_0 \approx \mathbb{W}_p k[\![u_1, \dots, u_{n-1}]\!]$.

$\Longrightarrow \coprod \mathrm{Spec} A_r$ is a "graded affine category scheme".
$M \in \mathrm{QCoh}(\mathrm{Def})$ are same as A-comodules:

$$(\psi_r)\colon M \to \textstyle\prod_{r\geq 0} A_r \otimes_{A_0} M \quad \text{such that } \dots .$$

5. Morava E-theory

5.1. **Theorem** (Morava, Goerss-Hopkins-Miller, Strickland)**.** *There exists essentially unique comm S-algebra $E = E_{G_0/k}$ such that*

$$A_r[u, u^{-1}] \approx E^*(B\Sigma_{p^r})/I, \qquad |u| = 2$$

where $I =$ sum of images of transfers along all $\Sigma_i \times \Sigma_{p^r-i} \subset \Sigma_{p^r}$, $0 < i < p^r$.
In particular, $\pi_* E = A_0[u, u^{-1}]$.

6. Power operations for $K(n)$-local commutative E-algebras

$R =$ comm E-algebra: power operation

$$P_m\colon \pi_0 R \to \pi_0 R^{B\Sigma_m^+} \approx \pi_0 R \otimes_{E_0} E^0 B\Sigma_m.$$

(Iso uses R is $K(n)$-local.)
Obtain *ring homomorphims*

$$\psi_r\colon \pi_0 R \to \pi_0 R \otimes_{E_0} E^0 B\Sigma_{p^r} \to \pi_0 R \otimes_{A_0} A_r.$$

This makes $\pi_0 R$ into A-comodule. Hence, we have

$$\pi_0\colon \mathrm{Alg}(E)_{K(n)} \to \mathrm{QCoh}(\mathrm{Def}).$$

6.1. **Proposition.** *Exists $\mathcal{A} = \mathcal{A}_{G_0}$, monadic over complete E_*-modules, and lift*

$$\begin{tikzcd} & \mathcal{A} \arrow[r] \arrow[d] & \mathrm{QCoh}(\mathrm{Def}, \mathrm{Ring}^*)_{\mathrm{Frob}} \arrow[dl] \\ \mathrm{Alg}(E)_{K(n)} \arrow[ur, dotted, "\pi_*"] \arrow[r, "\pi_*"'] & \mathrm{Mod}(E_*) & \end{tikzcd}$$

Forget factors through $\mathcal{A} \to \mathrm{QCoh}(\mathrm{Def}, \mathrm{Ring}^*)_{\mathrm{Frob}}$ (graded quasicoherent sheaves of (complete) commutative rings on Def *which satisfy a "Frobenius congruence"). Restricts to equivalence*

$$\mathcal{A}^{\mathrm{tf}} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Def}, \mathrm{Ring}^*)^{\mathrm{tf}}_{\mathrm{Frob}},$$

of full subcategories of p-torsion free objects.

(Ando-Hopkins-Strickland, R., Barthel-Frankland.)

Frobenius congruence. *Skip?* $R \in \mathrm{QCoh}(\mathrm{Def}, \mathrm{Ring})$ such that for $A \supseteq \mathbb{F}_p$,

$$A^{\phi} \otimes_A R_A(G, (i, \alpha)) \xrightarrow{\sim} R_A(\phi^* G, \phi^*(i, \alpha)) = R_A(\phi^* G, F_*(i, \alpha)) \xrightarrow{F^*} R_A(G, (i, \alpha))$$

coincides with relative Frobenius on ring $R_A(G, (i, \alpha))$.

Example. $G_0 = \widehat{\mathbb{G}}_m/\mathbb{F}_p$, $E = KU_p$. All $A_r = \mathbb{Z}_p$.

$\mathcal{A} \approx$ category of p-complete $\mathbb{Z}/2$-graded θ^p-ring (Bousfield).

A θ^p-ring (non-graded) is commutative ring A with function $\theta\colon A \to A$ such that

$$\theta(0) = 0, \qquad \theta(x + y) = \theta(x) + \theta(y) - \tfrac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} x^k y^{p-k},$$

$$\theta(xy) = x^p \theta(y) + y^p \theta(x) + p\theta(x)\theta(y).$$

The map $\psi(x) := x^p + p\theta(x)$ is ring homomorphism, giving "coaction" $M \to A_1 \otimes_{A_0} M = M$.

7. Quadratic description of QCoh(Def)

Recall that QCoh(Def) are comodules for $\{A_r\}$.

7.1. **Proposition.** *The structure of comodule on is completely determined by an A_0-module M, together with A_0-module map*

$$\psi\colon M \to {}^t A_1{}^s \otimes_{A_0} M$$

such that there exists a dotted arrow A_0-module map in

$$\begin{tikzcd} M \arrow[rr, "\psi"] \arrow[d, dashed] & & {}^t A_1{}^s \otimes_{A_0} M \arrow[d, "\mathrm{id} \otimes \psi"] \\ {}^t A_2{}^s \otimes_{A_0} M \arrow[r, "\nabla \otimes \mathrm{id}"'] & {}^t A_1{}^s \otimes_{A_0} {}^t A_1{}^s \otimes_{A_0} M \arrow[r, phantom, ""] & {} \end{tikzcd}$$

(Note $\nabla \otimes \mathrm{id}$ is always mono.)

Thus, a small amount of data $(A_1, s, t, A_2 \subset A_1 \otimes A_1)$ describes the category QCoh(Def).

7.2. *Remark. Skip?* At height 2, have $w\colon A_1 \to A_1$ ring homomorphism classifying "dual isogeny". Whence isomorphism $(A_1 \otimes_{A_0} A_1)/\nabla(A_2) \approx A_1/s(A_0)$ of A_0-bimodules, using $w \times \mathrm{id}\colon A_1 \otimes_{A_0} A_1 \to A_1$. Condition on ψ is $(w \times \psi)\psi \equiv 0 \mod s(A_0)$.

At height 2, small primes, this has been worked out explicitly (R., Zhu).

7.3. *Remark. Skip?* For a s.s. curve over $\mathbb{F}_2$, have:

$$A_0 = \mathbb{Z}_2[\![a]\!], \quad A_1 = A_0[d]/(d^3 - ad - 2),$$

$$s(a) = a, \qquad t(a) = w(a) = a^2 + 3d - ad^2, \qquad w(d) = a - d^2.$$

At all primes at height 2, can describe everything mod p.

Example: ht 2, any p. *Skip?* $G_0/F_p =$ completion of particular s.s. curve. Then

$$A_0/p \approx \mathbb{F}_p[\![a]\!], \qquad A_1/p \approx \mathbb{F}_p[\![a_0, a_1]\!]/\big((a_0^p - a_1)(a_0 - a_1^p)\big),$$

$$A_2/p \approx \mathbb{F}_p[\![a_0, a_1]\!]/\big((a_0^{p^2} - a_1)(a_0^p - a_1^p)(a_0 - a_1^{p^2})\big).$$

$$s\colon a \mapsto a_0, \quad t\colon a \mapsto a_1, \quad \nabla\colon a_0, a_2 \mapsto 1 \otimes a_0, a_1 \otimes 1.$$

Koszul. QCoh(Def) has finite homological dimension $2n$, and comes with "functorial small resolutions". Assuming we have data as above, we can compute Ext.

Skip? At height 2, $\mathrm{Ext}_{\mathrm{QCoh(Def)}}(M,N)$ for M projective A_0-module is H_* of

$$\mathrm{Hom}_{A_0}(M,N) \to \mathrm{Hom}_{A_0}(M, {}^{t}A_1{}^{s} \otimes_{A_0} N) \to \mathrm{Hom}_{A_0}(M, {}^{w^2 s}(A_1/sA_0)^{s} \otimes_{A_0} N)$$
$$f \mapsto \psi_N f - (\mathrm{id} \otimes f)\psi_M, \qquad g \mapsto (w \times \psi_N)g + (w \times g)\psi_M.$$

8. Spectral sequence for maps in $\mathrm{Alg}(E)_{K(n)}/E$

Let R, F augmented $K(n)$-local E-algebras. $\Longrightarrow$ spectral sequence

$$E_2^{s,t} \Longrightarrow \pi_{t-s}\mathrm{Alg}(E)_{/E}(R,F).$$

For π_*R *smooth* as a (complete) π_*E-algebra, and π_*R and π_*F concentrated in even degrees,

$$E_2^{s,t} = \begin{cases} \mathcal{A}(\pi_*R, \pi_*F) & (s,t) = (0,0), \\ \mathrm{Ext}^s_{\mathrm{QCoh(Def)}}(\omega^{-1} \otimes \widehat{Q}(\pi_*R), \omega^{t/2-1} \otimes \pi_*\overline{F}) & \text{otherwise.} \end{cases}$$

$\widehat{Q}$ is (completion of) indecomposables; $\pi_*\overline{F} \subset \pi_*F$ is augmentation ideal.

Example. (Special case of conjecture[1] of Hopkins-Lurie.)

Fix $G_0/\overline{\mathbb{F}}_p$ over alg closed field, height 2. (E.g., completion of a supersingular elliptic curve.)

Can show

$$\mathrm{Alg}(S)(\Sigma^\infty_+\mathbb{Z}, E) \approx \overline{\mathbb{F}}_p^\times \times K(\mathbb{Z}_p, 3).$$

(Same as $\mathrm{Alg}(E)_{/E}\big((E \wedge \Sigma^\infty_+\mathbb{Z})_{K(n)}, E \times E\big)$.)

This is less exciting than it looks: know $\pi_{*\geq 4} = 0$ by Ravenel-Wilson, and π_3 is known (e.g., Sati-Westerland).

Proof. Have that $\widehat{Q}(E_*^\wedge\mathbb{Z}) \approx \deg$. Calculate explicitly, using explicit height 2 formulas. All $E_2^{s,t}$ vanish *except* $E_2^{0,0} \approx \overline{\mathbb{F}}_p^\times$ and

$$E_2^{1,4} = \mathrm{Ext}^1_{\mathrm{QCoh(Def)}}(\omega^{-1} \otimes \deg, \omega) \approx \mathbb{Z}_p.$$

Remark. Assume G_0 is from s.s. elliptic curve C_0. $E_2^{1,4}$ generated by Hodge class:

$$0 \to \omega \to H^1_{\mathrm{dR}}(C/S) \to \omega^{-1} \otimes \deg \to 0,$$

of universal deformation $C/\mathrm{Spec}(A_0)$.

Remark.

$$\pi_3\mathrm{Alg}(S)(\Sigma^\infty_+\mathbb{Z}, \mathrm{TMF}_p) = [\Sigma^3 H\mathbb{Z}, \mathrm{gl}_1(\mathrm{TMF}_p)] \approx \mathbb{Z}_p^{c_p},$$

(p-complete TMF.)

$c_p = \dim \mathrm{MF}_2(\Gamma_0(p))^{W=-1} =$ (s.s. j-invts in $\mathbb{F}_p$) $+ \frac{1}{2}$(s.s. j-invts in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$). (Ogg.)

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[1] Word on the street: theorem.