Mike Mandell: E_n genera

The idea of this project is to try to understand: which cobordism invariants are realized as maps of structured ring spectra? This is based on `arXiv:1310.3336`, and is all joint with Greg Chadwick.

Recall that a *genus* is a cobordism invariant. Generally, we want this to come as a map $MG \to R$ for G some structure group – hopefully a map of ring spectra; the manifold invariants come after applying π_*. We'll be most interestd in MSO and MU (orientable and stably almost-complex manifolds, resp.). Recall that e.g. $MU_*(X)$ is the cobordism theory of stably almost-complex manifolds equipped with a map to some parametrizing space X.

Now, MSO and MU are *commutative S-algebras*, a/k/a "E_∞-ring spectra". So, a natural question is: Which genera come from maps of commutative S-algebras? Or easier, how about just S-algebras, a/k/a "A_∞-ring spectra", a/k/a E_1-ring spectra? Intermediately, we also can ask about E_n for all n. This will be our focus.

Here is the main result.

Theorem 6. *Let R be an E_2-ring spectrum with no odd homotopy (i.e., it's "even"). Then any map of (homotopy) ring spectra $MU \to R$ lifts to a map of E_2-ring spectra. If moreover $2^{-1} \in \pi_0 R$, then any map of ring spectra $MSO \to R$ lifts to a map of E_2-ring spectra.*

Corollary 2. *The Quillen idempotent $MU \to MU$ is an E_2 map, and taking telescopes yields that BP is also an E_2-ring spectrum.*

This is independent of Basterra–Mandell, who show that BP is E_4 but didn't answer the question of the Quillen idempotent.

Corollary 3. *Given source and target ring spectra where the target is E_2, any map of ring spectra can be lifted to an E_2 map.*

This all runs through the Pontryagin–Thom theorem, which says that cobordism theories are represented by Thom spectra, and that in this framework genera are precisely multiplicative orientations.

Definition 10. An *R-orientation* for $\mathbb{C}^n$-vector bundles is an element of $R^{2n}(EU(n), \tilde{E}U(n))$ that restricts to a generator of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - 0)$ on every fiber of $BU(n)$. Here, $EU(n)$ is the universal $\mathbb{C}^n$-vector bundle and $\tilde{E}U(n)$ is obtained by deleting the zero-section. An orientation is *multiplicative* if we actually get a unit element of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - 0) \cong R^0(*)$. By excision, an orientation is equivalently an element of $\tilde{R}^{2n}(TU(n))$, for $TU(n)$ the Thom space of $EU(n)$. We can actually define $MU = \operatorname{colim} \Sigma^\infty_{-2n} TU(n)$, from which we get that $[MU, R] \cong \operatorname{colim} \tilde{R}^{2n}(TU(n))$.

Now, the *Thom diagonal* is a map $MU \to MU \wedge BU_+$, coming from maps $TU(n) \to TU(n) \wedge BU(n)_+$; this gives an action $R^*MU \otimes R^*BU \to R^*MU$. For a fixed orientation $(MU \to R) \in R^*MU$, this yields a map $R^*BU \to R^*MU$. This is precisely the *Thom isomorphism*.

Now, Mahowald proved that the Thom diagonal is actually a map of ring spectra, and Lewis proved that actually this is even an E_∞ map.

Hence, for $\sigma : MU \to R$ a multiplicative orientation, a map $BU \to R$ – i.e., a map $\Sigma^\infty_+ BU \to R$ of spectra – gives a ring map

$$MU \to MU \wedge BU_+ \to R \wedge R \to R.$$

Now, maps of *ring* spectra $\Sigma^\infty_+ BU \to R$ are the same as h-space maps $BU \to \Omega^\infty R^\times$, and these give *ring* maps $MU \to R$. We view this as $R^0 BU \to R^0 MU$. We can state Quillen's theorem in this light.

Theorem 7. *Ring maps $MU \to R$ are exactly the elements of R^0MU that correspond to elements of $\tilde{R}^2(TU(1))$ that restrict to the unit element of $\tilde{R}^2(S^2)$.*

Now, assume that R is an E_n-ring spectrum, and suppose that $\sigma : MU \to R$ is E_n. We can play the same game, and we get the same comparison: an equivalence between E_n maps $\Sigma^\infty_+ BU \to R$ and E_n maps $BU \to \Omega^\infty R^\times$. Moreover, Dunn proved that if R is E_n then $R \wedge R \to R$ is E_{n-1}...but we're interested in E_n maps, so we just bump up n by 1. That is, if R is E_{n+1} then we have a natural action of $\operatorname{map}_{E_n}(BU, \Omega^\infty R^\times)$ on $\operatorname{map}_{E_n}(MU, R)$. Assuming the target is nonempty, choosing a basepoint yields a map $\operatorname{map}_{E_n}(BU, \Omega^\infty R^\times) \to \operatorname{map}_{E_n}(MU, R)$.

So, assume that R is E_{n+1} and that $\sigma : MU \to R$ is E_n.

Theorem 8. *The Thom map induces a map*

$$\mathrm{map}_{E_n}(BU, \Omega^\infty R^\times) \simeq \mathrm{map}_{E_n}(\Sigma^\infty_+ BU, R) \to \mathrm{map}_{E_n}(MU, R),$$

and this is a weak equivalence.

One might call this a *multiplicative Thom isomorphism*. Whereas one proves the usual Thom isomorphism cell by cell, here it works slightly better to assume R is connective (which is fine since MU is too) and look at its *multiplicative* Postnikov tower.

To describe these sorts of Postnikov towers, suppose R is a connective E_n-ring spectrum and let $Z = \pi_0 R$. Then we get a Postnikov tower

$$R \to \cdots R\langle 2\rangle \to R\langle 1\rangle \to R\langle 0\rangle = HZ$$

in the category of E_n-ring spectra. The key – due to Kriz – is that this is actually a tower of *principal* fibrations of E_n-ring spectra, given by pullbacks

$$\begin{array}{ccc} R\langle j+1\rangle & \longrightarrow & HZ \\ \downarrow & & \downarrow \\ R\langle j\rangle & \xrightarrow[k_{j+1}]{} & HZ \vee \Sigma^{j+2} H(\pi_{j+1} R) \end{array}$$

in E_n-ring spectra (where the bottom-right is the square-zero ring spectrum, and is in fact E_∞). This is basically just a freed-up version of the spectrum version where we remove both instances of HZ.

Now, we define *topological Quillen cohomology* (relative to Z) by $H^*_{E_n}(A; M) = \pi_0 \mathrm{map}_{E_n/HZ}(A, HZ \vee \Sigma^* HM)$. This gives rise to an obstruction theory: there's an obstruction in $H^{j+2}_{E_n}(A; \pi_{j+1} R)$ to lifting an E_n-ring map $A \to R\langle j\rangle$ to an E_n-ring map $A \to R\langle j+1\rangle$. (This comes from mapping A into the pullback square above.) The space of lifts is either empty or is a "free orbit" on the topological monoid

$$\mathrm{map}_{E_n/HZ}(A, HZ \vee \Sigma^{j+1} HM) \simeq \Omega \mathrm{map}_{E_n/HZ}(A, HZ \vee \Sigma^{j+2} HM) \simeq \Omega^2 \mathrm{map}_{E_n/HZ}(A, HZ \vee \Sigma^{j+3} HM) \simeq \cdots .$$

This gives a sort of Atiyah–Hirzebruch spectral sequence: For E_n-ring spectra A and R (with A satisfying mild hypotheses: maybe connective and also with a chosen map $\pi_0 A \to Z$), there is an "obstructed" spectral sequence

$$E^2_{p,q} = H^p_{E_n}(A; \pi_q R) \Rightarrow \pi_{q-p} \mathrm{map}_{E_n}(A, R).$$

(This is "obstructed" in the sense that you can only pass to the next page when the obstructions vanish.)

Now, let's say some stuff! Chadwick's thesis handles the multiplicative Thom isomorphism in the case where the target is E_∞. This follows from what we've said, since the Thom map induces isomorphisms on E^2-terms of the obstructed spectral sequence, and hence an isomorphism of abutments.

Finally, we have the main result.

Theorem 9. *If R is an even E_3-ring spectrum, then $\mathrm{map}_{E_2}(MU, R) \simeq \mathrm{map}_{E_2}(BU, \Omega^\infty R^\times)$, and $\pi_{-*} \mathrm{map}_{E_2}(BU, \Omega^\infty R^\times) \to R^*(BU(1))$ is surjective. Thus, every map of ring spectra $MU \to R$ lifts to a map of E_2-ring spectra.*

For R only E_2-we have to be more careful, but in fact the theorem still holds.

E_n Genera

Michael A. Mandell

Indiana University

MSRI Workshop on
Reimagining the Foundations of Algebraic Topology

April 11, 2014

Overview

Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Preprint arXiv:1310.3336
- Builds on Greg's thesis: *Structured orientations of Thom spectra* (Thesis, Indiana University, 2012)

Overview

Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Preprint arXiv:1310.3336
- Builds on Greg's thesis: *Structured orientations of Thom spectra* (Thesis, Indiana University, 2012)

Outline

Overview

Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Preprint arXiv:1310.3336
- Builds on Greg's thesis: *Structured orientations of Thom spectra* (Thesis, Indiana University, 2012)

Outline

1. Introduction and main result

Overview

Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Preprint arXiv:1310.3336
- Builds on Greg's thesis: *Structured orientations of Thom spectra* (Thesis, Indiana University, 2012)

Outline

1. Introduction and main result
2. Genera and orientations

Overview

Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Preprint arXiv:1310.3336
- Builds on Greg's thesis: *Structured orientations of Thom spectra* (Thesis, Indiana University, 2012)

Outline

1. Introduction and main result
2. Genera and orientations
3. Multiplicative Thom isomorphism

Overview

Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Preprint arXiv:1310.3336
- Builds on Greg's thesis: *Structured orientations of Thom spectra* (Thesis, Indiana University, 2012)

Outline

1. Introduction and main result
2. Genera and orientations
3. Multiplicative Thom isomorphism
4. Topological Quillen cohomology

Overview

Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Preprint arXiv:1310.3336
- Builds on Greg's thesis: *Structured orientations of Thom spectra* (Thesis, Indiana University, 2012)

Outline

1. Introduction and main result
2. Genera and orientations
3. Multiplicative Thom isomorphism
4. Topological Quillen cohomology and unstable obstructed Atiyah-Hirzebruch spectral sequences

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of an abelian group A

$$M^m \mapsto \gamma(M) \in A$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$.

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of an abelian group A

$$M^m \mapsto \gamma(M) \in A$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$.

Extra structure: (e.g.)

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of an abelian group A

$$M^m \mapsto \gamma(M) \in A$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$.

Extra structure: (e.g.)

- Oriented manifolds (MSO_*)

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

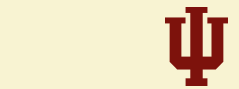
It assigns to every manifold (with extra structure) an element of an abelian group A

$$M^m \mapsto \gamma(M) \in A$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$.

Extra structure: (e.g.)

- Oriented manifolds (MSO_*)
- Stably almost complex manifolds (MU_*)



Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

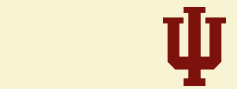
It assigns to every manifold (with extra structure) an element of a graded abelian group A_*

$$M^m \mapsto \gamma(M) \in A_m$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$.

Extra structure: (e.g.)

- Oriented manifolds (MSO_*)
- Stably almost complex manifolds (MU_*)



Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of a graded ring R_*

$$M^m \mapsto \gamma(M) \in R_m$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$.

Extra structure: (e.g.)

- Oriented manifolds (MSO_*)
- Stably almost complex manifolds (MU_*)

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of a graded ring R_*

$$M^m \mapsto \gamma(M) \in R_m$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$ and for any M^m, N^n,

$$\gamma(M \times N) = \gamma(M) \cdot \gamma(N) \in R_{m+n}$$

Extra structure: (e.g.)

- Oriented manifolds (MSO_*)
- Stably almost complex manifolds (MU_*)

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of a graded ring R_*

$$M^m \mapsto \gamma(M) \in R_m$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$ and for any M^m, N^n,

$$\gamma(M \times N) = \gamma(M) \cdot \gamma(N) \in R_{m+n}$$

Extra structure: (e.g.)

- Oriented manifolds with map to X $\quad (MSO_*(X))$
- Stably almost complex manifolds with map to X $\quad (MU_*(X))$

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of homology theory R_*

$$M^m \mapsto \gamma(M) \in R_m(X)$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$ and for any M^m, N^n,

$$\gamma(M \times N) = \gamma(M) \cdot \gamma(N) \in R_{m+n}$$

Extra structure: (e.g.)

- Oriented manifolds with map to X $\quad (MSO_*(X))$
- Stably almost complex manifolds with map to X $\quad (MU_*(X))$

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of a multiplicative homology theory R_*

$$M^m \mapsto \gamma(M) \in R_m(X)$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$ and for any M^m, N^n,

$$\gamma(M \times N) = \gamma(M) \cdot \gamma(N) \in R_{m+n}$$

Extra structure: (e.g.)

- Oriented manifolds with map to X $\quad (MSO_*(X))$
- Stably almost complex manifolds with map to X $\quad (MU_*(X))$

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of a multiplicative homology theory R_*

$$M^m \mapsto \gamma(M) \in R_m(X)$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$ and for any M^m, N^n,

$$\gamma(M \times N) = \gamma(M) \cdot \gamma(N) \in R_{m+n}(X \times Y)$$

Extra structure: (e.g.)

- Oriented manifolds with map to X $\quad (MSO_*(X))$
- Stably almost complex manifolds with map to X $\quad (MU_*(X))$

Review of Genera

Definition

A

It a
mu

su
for

A genus is a map of multiplicative homology theories

$$MSO_* \to R_* \quad \text{or} \quad MU_* \to R_*$$

Extra structure: (e.g.)

- Oriented manifolds with map to X $(MSO_*(X))$
- Stably almost complex manifolds with map to X $(MU_*(X))$

Review of Genera

Definition

A

It a
mu

su
for

A genus is a map of multiplicative homology theories

$$MSO_* \to R_* \quad \text{or} \quad MU_* \to R_*$$

or, better, a map of ring spectra

$$MSO \to R \quad \text{or} \quad MU \to R$$

Extra structure: (e.g.)

- Oriented manifolds with map to X $\quad (MSO_*(X))$
- Stably almost complex manifolds with map to X $\quad (MU_*(X))$

Structured Genera

Genera are maps of ring spectra $MSO \to R$, $MU \to R$.

Structured Genera

Genera are maps of ring spectra $MSO \to R$, $MU \to R$.

Question

Which genera come from maps of “highly structured” ring spectra?

Structured Genera

Genera are maps of ring spectra $MSO \to R$, $MU \to R$.

Question

Which genera come from maps of "highly structured" ring spectra?

MSO and MU are commutative S-algebras (E_∞ ring spectra).

Structured Genera

Genera are maps of ring spectra $MSO \to R$, $MU \to R$.

Question

Which genera come from maps of "highly structured" ring spectra?

MSO and MU are commutative S-algebras (E_∞ ring spectra).

Which genera come from

- Maps of commutative S-algebras = E_∞ ring spectra ?

Structured Genera

Genera are maps of ring spectra $MSO \to R$, $MU \to R$.

Question

Which genera come from maps of "highly structured" ring spectra?

MSO and MU are commutative S-algebras (E_∞ ring spectra).

Which genera come from

- Maps of commutative S-algebras = E_∞ ring spectra
- Maps of S-algebras = A_∞ ring spectra ?

Structured Genera

Genera are maps of ring spectra $MSO \to R$, $MU \to R$.

Question

Which genera come from maps of "highly structured" ring spectra?

MSO and *MU* are commutative S-algebras (E_∞ ring spectra).

Which genera come from

- Maps of commutative S-algebras = E_∞ ring spectra
- Maps of S-algebras = A_∞ ring spectra = E_1 ring spectra ?

Structured Genera

Genera are maps of ring spectra $MSO \to R$, $MU \to R$.

Question

Which genera come from maps of "highly structured" ring spectra?

MSO and *MU* are commutative S-algebras (E_∞ ring spectra).

Which genera come from

- Maps of commutative S-algebras = E_∞ ring spectra
- Maps of S-algebras = A_∞ ring spectra = E_1 ring spectra
- Maps of E_n ring spectra ?

Main Result

Theorem

Let R be an E_2 ring spectrum with $\pi_n R = 0$ for all n odd.

Main Result

Theorem

Let R be an E_2 ring spectrum with $\pi_n R = 0$ for all n odd.

Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

Main Result

Theorem

Let R be an E_2 ring spectrum with $\pi_n R = 0$ for all n odd.

Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

If $\frac{1}{2} \in \pi_0 R$, then any map of ring spectra $MSO \to R$ lifts to a map of E_2 ring spectra.

Main Result

Theorem

Let R be an E_2 ring spectrum with $\pi_n R = 0$ for all n odd.

Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

If $\frac{1}{2} \in \pi_0 R$, then any map of ring spectra $MSO \to R$ lifts to a map of E_2 ring spectra.

Consequence: Taking $R = MU$, the Quillen idempotent is an E_2 map, and BP is an E_2 ring spectrum (Chadwick's IU PhD Thesis)

Independent of [Basterra-Mandell] which showed that BP is E_4 but left open the question of Quillen idempotent

Main Result

Theorem

Let R be an E_∞ ring spectrum with $\pi_n R = 0$ for all n odd.
If there exits an E_∞ ring map $MU \to R$
Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

Consequence: Taking $R = MU$, the Quillen idempotent is an E_2 map, and BP is an E_2 ring spectrum (Chadwick's IU PhD Thesis)

Independent of [Basterra-Mandell] which showed that BP is E_4 but left open the question of Quillen idempotent

Main Result

Theorem

Let R be an E_∞ ring spectrum with $\pi_n R = 0$ for all n odd.
If there exits an E_∞ ring map $MU \to R$
Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

Consequence: Taking $R = MU$, the Quillen idempotent is an E_2 map, and BP is an E_2 ring spectrum (Chadwick's IU PhD Thesis)

Independent of [Basterra-Mandell] which showed that BP is E_4
but left open the question of Quillen idempotent

Main Result

Theorem

Let R be an E_∞ ring spectrum with $\pi_n R = 0$ for all n odd.
If there exits an E_∞ ring map $MU \to R$
Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

Consequence: Taking $R = MU$, the Quillen idempotent is an E_2 map, and BP is an E_2 ring spectrum (Chadwick's IU PhD Thesis)

Independent of [Basterra-Mandell] which showed that BP is E_4 but left open the question of Quillen idempotent

Main Result

Theorem

Let R be an E_2 ring spectrum with $\pi_n R = 0$ for all n odd.

Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

If $\frac{1}{2} \in \pi_0 R$, then any map of ring spectra $MSO \to R$ lifts to a map of E_2 ring spectra.

Main Result

Theorem

Let R be an E_2 ring spectrum with $\pi_n R = 0$ for all n odd.

Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

If $\frac{1}{2} \in \pi_0 R$, then any map of ring spectra $MSO \to R$ lifts to a map of E_2 ring spectra.

Consequence: Under the hypotheses above any map of ring spectra lifts to a map of S-algebras (A_∞ ring spectra).

Main Result

Theorem

Let R be an E_2 ring spectrum with $\pi_n R = 0$ for all n odd.

Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

If $\frac{1}{2} \in \pi_0 R$, then any map of ring spectra $MSO \to R$ lifts to a map of E_2 ring spectra.

Consequence: Under the hypotheses above any map of ring spectra lifts to a map of S-algebras (A_∞ ring spectra).

Proof: Calculate π_* of the space of E_2 maps and look at the map to π_* of the space of ring spectra maps.

Main Result

Theorem

Let R be an E_2 ring spectrum with $\pi_n R = 0$ for all n odd.

Then any map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

If $\frac{1}{2} \in \pi_0 R$, then any map of ring spectra $MSO \to R$ lifts to a map of E_2 ring spectra.

Consequence: Under the hypotheses above any map of ring spectra lifts to a map of S-algebras (A_∞ ring spectra).

Proof: Calculate π_0 of the space of E_2 maps and look at the map to π_0 of the space of ring spectra maps.

Genera and Orientations

Pontryagin-Thom theorem

- Cobordism theories are represented by Thom spectra
- Genera are multiplicative orientations

Genera and Orientations

Pontryagin-Thom theorem

- Cobordism theories are represented by Thom spectra
- Genera are multiplicative orientations

Genera and Orientations

Pontryagin-Thom theorem

- Cobordism theories are represented by Thom spectra
- Genera are multiplicative orientations

Genera and Orientations

Pontryagin-Thom theorem

- Cobordism theories are represented by Thom spectra
- Genera are multiplicative orientations

$\rightsquigarrow BU(n)$ classifying space for $\mathbb{C}^n$-vector bundles

$\rightsquigarrow PU(n)$ total space of universal principal bundle
(free contractible CW $U(n)$-space)

$\rightsquigarrow EU(n) = PU(n) \times_{U(n)} \mathbb{C}^n$ total space of universal vector bundle

$\rightsquigarrow \mathring{E}U(n) = PU(n) \times_{U(n)} (\mathbb{C}^n - \{0\})$

$TU(n) = PU(n)_+ \wedge_{U(n)} S^{2n}$ Thom space

Genera and Orientations

Pontryagin-Thom theorem

- Cobordism theories are represented by Thom spectra
- Genera are multiplicative orientations

$BU(n)$ classifying space for $\mathbb{C}^n$-vector bundles

$PU(n)$ total space of universal principal bundle (free contractible CW $U(n)$-space)

$EU(n) = PU(n) \times_{U(n)} \mathbb{C}^n$ total space of universal vector bundle

$\mathring{E}U(n) = PU(n) \times_{U(n)} (\mathbb{C}^n - \{0\})$

$TU(n) = PU(n)_+ \wedge_{U(n)} S^{2n}$ Thom space

Genera and Orientations

Pontryagin-Thom theorem

- Cobordism theories are represented by Thom spectra
- Genera are multiplicative orientations

$BU(n)$ classifying space for $\mathbb{C}^n$-vector bundles

$PU(n)$ total space of universal principal bundle (free contractible CW $U(n)$-space)

$EU(n) = PU(n) \times_{U(n)} \mathbb{C}^n$ total space of universal vector bundle

$\mathring{E}U(n) = PU(n) \times_{U(n)} (\mathbb{C}^n - \{0\})$

$TU(n) = PU(n)_+ \wedge_{U(n)} S^{2n}$ Thom space

Genera and Orientations

Pontryagin-Thom theorem

- Cobordism theories are represented by
- Genera are multiplicative orientations

$BU(n)$ classifying space for $\mathbb{C}^n$-vector bundles

$PU(n)$ total space of universal principal bundle (free contractible CW $U(n)$-space)

$EU(n) = PU(n) \times_{U(n)} \mathbb{C}^n$ total space of universal vector bundle

$\mathring{E}U(n) = PU(n) \times_{U(n)} (\mathbb{C}^n - \{0\})$

$TU(n) = PU(n)_+ \wedge_{U(n)} S^{2n}$ Thom space

Genera and Orientations

$BU(n)$ classifying space for $\mathbb{C}^n$-vector bundles

$PU(n)$ total space of universal principal bundle (free contractible CW $U(n)$-space)

$EU(n) = PU(n) \times_{U(n)} \mathbb{C}^n$ total space of universal vector bundle

$\mathring{E}U(n) = PU(n) \times_{U(n)} (\mathbb{C}^n - \{0\})$

$TU(n) = PU(n)_+ \wedge_{U(n)} S^{2n}$ Thom space

Pontryagin-Thom theorem

- Cobordism theories are represented by
- Genera are multiplicative orientations

An R-orientation for $\mathbb{C}^n$-vector bundles is an element of $R^{2n}(EU(n), \mathring{E}U(n))$ that restricts to a generator of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$ on each fiber (of $BU(n)$).

Multiplicative:

- Restricts to unit element of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$
- $R^{2m}(EU(m), \mathring{E}U(m)) \otimes R^{2n}(EU(n), \mathring{E}U(n))$
  $\to R^{2m+2n}((EU(m), \mathring{E}U(m)) \times (EU(n), \mathring{E}U(n)))$
  $\to R^{2m+2n}(EU(m+n), \mathring{E}U(m+n))$

Excision: $R^{2n}(EU(n), \mathring{E}U(n)) \cong \tilde{R}^{2n}(TU(n))$.

$MU = \operatorname{colim} \Sigma^{\infty}_{-2n} TU(n)$ $\qquad [MU, R] = R^0(MU) = \lim \tilde{R}^{2n}(TU(n))$

Genera and Orientations

$BU(n)$ classifying space for $\mathbb{C}^n$-vector bundles

$PU(n)$ total space of universal principal bundle (free contractible CW $U(n)$-space)

$EU(n) = PU(n) \times_{U(n)} \mathbb{C}^n$ total space of universal vector bundle

$\mathring{E}U(n) = PU(n) \times_{U(n)} (\mathbb{C}^n - \{0\})$

$TU(n) = PU(n)_+ \wedge_{U(n)} S^{2n}$ Thom space

Pontryagin-Thom theorem

- Cobordism theories are represented by
- Genera are multiplicative orientations

An R-orientation for $\mathbb{C}^n$-vector bundles is an element of $R^{2n}(EU(n), \mathring{E}U(n))$ that restricts to a generator of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$ on each fiber (of $BU(n)$).

Multiplicative:

- Restricts to unit element of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\}) \cong R^0(*)$
- $R^{2m}(EU(m), \mathring{E}U(m)) \otimes R^{2n}(EU(n), \mathring{E}U(n))$
  $\to R^{2m+2n}((EU(m), \mathring{E}U(m)) \times (EU(n), \mathring{E}U(n)))$
  $\to R^{2m+2n}(EU(m+n), \mathring{E}U(m+n))$

$BU(m) \times BU(n) \downarrow BU(m+n)$

Excision: $R^{2n}(EU(n), \mathring{E}U(n)) \cong \tilde{R}^{2n}(TU(n))$.

$MU = \operatorname{colim} \Sigma^\infty_{-2n} TU(n)$ $\qquad [MU, R] = R^0(MU) = \lim \tilde{R}^{2n}(TU(n))$

Genera and Orientations

$BU(n)$ classifying space for $\mathbb{C}^n$-vector bundles

$PU(n)$ total space of universal principal bundle (free contractible CW $U(n)$-space)

$EU(n) = PU(n) \times_{U(n)} \mathbb{C}^n$ total space of universal vector bundle

$\mathring{E}U(n) = PU(n) \times_{U(n)} (\mathbb{C}^n - \{0\})$

$TU(n) = PU(n)_+ \wedge_{U(n)} S^{2n}$ Thom space

Pontryagin-Thom theorem

- Cobordism theories are represented by
- Genera are multiplicative orientations

An R-orientation for $\mathbb{C}^n$-vector bundles is an element of $R^{2n}(EU(n), \mathring{E}U(n))$ that restricts to a generator of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$ on each fiber (of $BU(n)$).

Multiplicative:

- Restricts to unit element of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$
- $R^{2m}(EU(m), \mathring{E}U(m)) \quad \otimes \quad R^{2n}(EU(n), \mathring{E}U(n))$
  $\to R^{2m+2n}((EU(m), \mathring{E}U(m)) \times (EU(n), \mathring{E}U(n)))$
  $\to R^{2m+2n}(EU(m+n), \mathring{E}U(m+n))$

Excision: $R^{2n}(EU(n), \mathring{E}U(n)) \cong \tilde{R}^{2n}(TU(n))$.

$MU = \operatorname{colim} \Sigma^{\infty}_{-2n} TU(n) \qquad [MU, R] = R^0(MU) = \lim \tilde{R}^{2n}(TU(n))$

Genera and Orientations

$BU(n)$ classifying space for $\mathbb{C}^n$-vector bundles

$PU(n)$ total space of universal principal bundle (free contractible CW $U(n)$-space)

$EU(n) = PU(n) \times_{U(n)} \mathbb{C}^n$ total space of universal vector bundle

$\mathring{E}U(n) = PU(n) \times_{U(n)} (\mathbb{C}^n - \{0\})$

$TU(n) = PU(n)_+ \wedge_{U(n)} S^{2n}$ Thom space

Pontryagin-Thom theorem

- Cobordism theories are represented by
- Genera are multiplicative orientations

An R-orientation for $\mathbb{C}^n$-vector bundles is an element of $R^{2n}(EU(n), \mathring{E}U(n))$ that restricts to a generator of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$ on each fiber (of $BU(n)$).

Multiplicative:

- Restricts to unit element of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$
- $R^{2m}(EU(m), \mathring{E}U(m)) \quad \otimes \quad R^{2n}(EU(n), \mathring{E}U(n))$
  $\to R^{2m+2n}((EU(m), \mathring{E}U(m)) \times (EU(n), \mathring{E}U(n)))$
  $\to R^{2m+2n}(EU(m+n), \mathring{E}U(m+n))$

Excision: $R^{2n}(EU(n), \mathring{E}U(n)) \cong \tilde{R}^{2n}(TU(n))$.

$MU = \operatorname{colim} \Sigma^{\infty}_{-2n} TU(n)$ $\qquad [MU, R] = R^0(MU) \cong \lim \tilde{R}^{2n}(TU(n))$

The Thom Isomorphism

Thom diagonal

$$MU \to MU \wedge BU_+$$

Gives an action

$$R^*(MU) \otimes R^*(BU) \to R^*(MU)$$

$$\left.\begin{array}{l} f\colon MU \to R \\ g\colon \Sigma^\infty_+ BU \to R \end{array}\right\} \quad \Longrightarrow \quad MU \to MU \wedge BU_+ \xrightarrow{f \wedge g} R \wedge R \to R$$

Taking f to be a fixed orientation, get a map

$$R^*BU \to R^*MU$$

Thom Isomorphism: This map is an isomorphism

The Thom Isomorphism

Thom diagonal

$$MU \to MU \wedge BU_+$$

Gives an action

$$R^*(MU) \otimes R^*(BU) \to R^*(MU)$$

$$\left.\begin{array}{l} f\colon MU \to R \\ g\colon \Sigma^\infty_+ BU \to R \end{array}\right\} \quad \Longrightarrow \quad MU \to MU \wedge BU_+ \xrightarrow{f \wedge g} R \wedge R \to R$$

Taking f to be a fixed orientation, get a map

$$R^* BU \to R^* MU$$

Thom Isomorphism: This map is an isomorphism

The Thom Isomorphism

Thom diagonal

$$MU \to MU \wedge BU_+$$

Gives an action

$$R^*(MU) \otimes R^*(BU) \to R^*(MU)$$

$$f\colon MU \to R$$
$$g\colon \Sigma^\infty_+ BU \to R$$

$$\Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{f\wedge g} R \wedge R \to R$$

Taking f to be a fixed orientation, get a map

$$R^*BU \to R^*MU$$

Thom Isomorphism: This map is an isomorphism

The Thom Isomorphism

Thom diagonal

$$MU \to MU \wedge BU_+$$

Gives an action

$$R^*(MU) \otimes R^*(BU) \to R^*(MU)$$

$$\left.\begin{array}{l} f\colon MU \to R \\ g\colon \Sigma^\infty_+ BU \to R \end{array}\right\} \quad \Longrightarrow \quad MU \to MU \wedge BU_+ \xrightarrow{f\wedge g} R \wedge R \to R$$

Taking f to be a fixed orientation, get a map

$$R^*BU \to R^*MU$$

Thom Isomorphism: This map is an isomorphism

The Thom Isomorphism

Thom diagonal

$$MU \to MU \wedge BU_+$$

Gives an action

$$R^*(MU) \otimes R^*(BU) \to R^*(MU)$$

$$\left.\begin{array}{l} f\colon MU \to R \\ g\colon \Sigma^\infty_+ BU \to R \end{array}\right\} \quad \Longrightarrow \quad MU \to MU \wedge BU_+ \xrightarrow{f \wedge g} R \wedge R \to R$$

Taking f to be a fixed orientation, get a map

$$R^*BU \to R^*MU$$

Thom Isomorphism: This map is an isomorphism

Multiplicative Structure

Thom diagonal

$$MU \to MU \wedge BU_+$$

is a map of ring spectra [Mahowald]

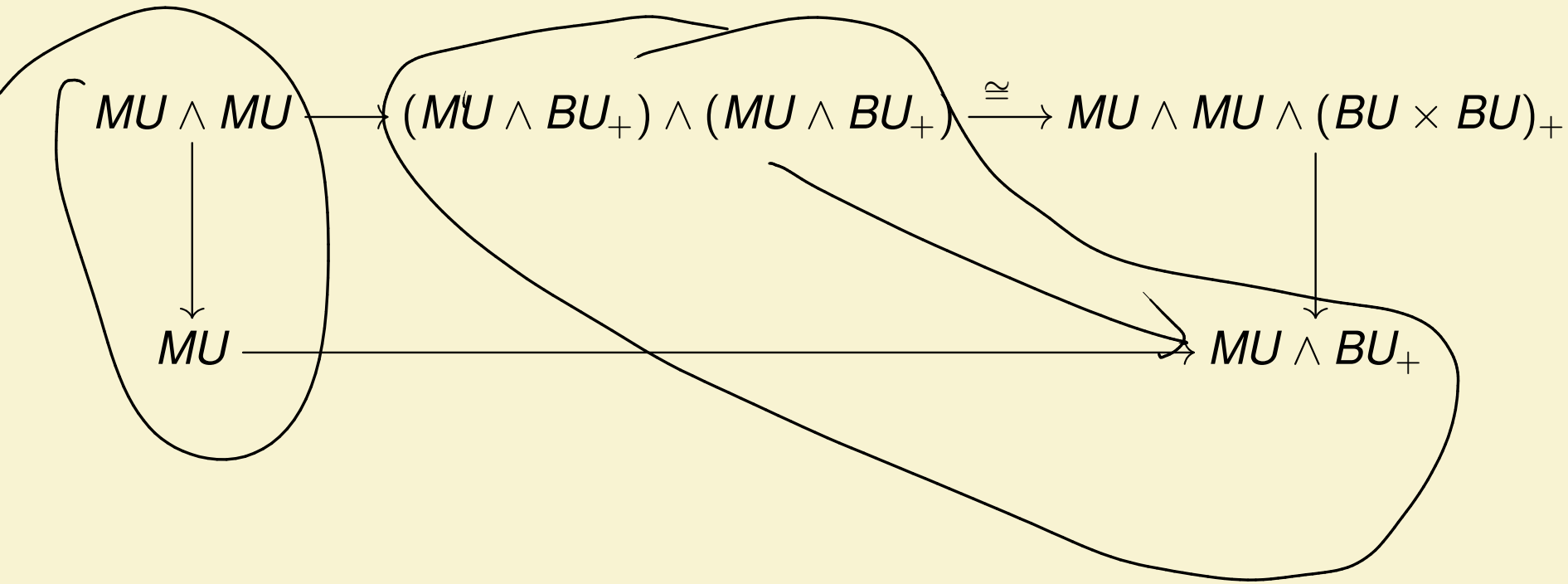
$$\begin{array}{ccccc} MU \wedge MU & \longrightarrow & (MU \wedge BU_+) \wedge (MU \wedge BU_+) & \xrightarrow{\cong} & MU \wedge MU \wedge (BU \times BU)_+ \\ \downarrow & & & & \downarrow \\ MU & & \longrightarrow & & MU \wedge BU_+ \end{array}$$

Multiplicative Structure

Thom diagonal

$$MU \to MU \wedge BU_+$$

is a map of ring spectra [Mahowald]



Multiplicative Structure

Thom diagonal

$$MU \to MU \wedge BU_+$$

is a map of ring spectra [Mahowald] in fact E_∞ ring spectra [Lewis]

$$\begin{array}{ccccc} MU \wedge MU & \longrightarrow & (MU \wedge BU_+) \wedge (MU \wedge BU_+) & \xrightarrow{\cong} & MU \wedge MU \wedge (BU \times BU)_+ \\ \downarrow & & & & \downarrow \\ MU & & \longrightarrow & & MU \wedge BU_+ \end{array}$$

Multiplicative Orientations

For a multiplicative orientation σ

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \quad \Longrightarrow \quad MU \to MU \wedge BU_+ \xrightarrow{\sigma\wedge g} R \wedge R \to R$$

takes

- Ring spectra maps $\Sigma^\infty_+ BU \to R$
  $= H$-space maps $BU \to \Omega^\infty R^\times$ } in $R^0(BU)$

to

- Ring spectra maps $MU \to R$ in $R^0(MU)$

Multiplicative Orientations

For a multiplicative orientation σ

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

takes

- Ring spectra maps $\Sigma^\infty_+ BU \to R$
  $=$ H-space maps $BU \to \Omega^\infty R^\times$

  $\Big\}$ in $R^0(BU)$

to

- Ring spectra maps $MU \to R$ in $R^0(MU)$

Multiplicative Orientations

For a multiplicative orientation σ

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma\wedge g} R \wedge R \to R$$

takes

- Ring spectra maps $\Sigma^\infty_+ BU \to R$
  $= H$-space maps $BU \to \Omega^\infty R^\times$
  $\Big\}$ in $R^0(BU)$

to

- Ring spectra maps $MU \to R$ in $R^0(MU)$

“Quillen’s Theorem”

For a multiplicative orientation, the Thom map takes maps of H-spaces in $[BU, \Omega^\infty R^\times] = R^0(BU)$ to maps of ring spectra in $[MU, R] = R^0(MU)$

Theorem (Hirzebruch, Dold, Quillen)

Maps of ring spectra in $R^0(MU)$ are in one-to-one correspondence with elements of $\tilde{R}^2(TU(1))$ that restrict to the unit element of $\tilde{R}^2(S^2)$

When a map of ring spectra $MU \to R$ exists, then:

- *The maps of ring spectra in $R^0(MU)$ are exactly the maps that correspond to maps of H-spaces in $R^0(BU)$ and*
- *Are in one-to-one correspondence with elements of $R^0(BU(1))$ via the map on R^0 induced by $BU(1) \to BU$.*

“Quillen’s Theorem”

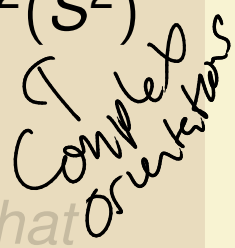
For a multiplicative orientation, the Thom map takes maps of H-spaces in $[BU, \Omega^\infty R^\times] = R^0(BU)$ to maps of ring spectra in $[MU, R] = R^0(MU)$

Theorem (Hirzebruch, Dold, Quillen)

Maps of ring spectra in $R^0(MU)$ are in one-to-one correspondence with elements of $\tilde{R}^2(TU(1))$ that restrict to the unit element of $\tilde{R}^2(S^2)$

When a map of ring spectra $MU \to R$ exists, then:

- *The maps of ring spectra in $R^0(MU)$ are exactly the maps that correspond to maps of H-spaces in $R^0(BU)$ and*
- *Are in one-to-one correspondence with elements of $R^0(BU(1))$ via the map on R^0 induced by $BU(1) \to BU$.*



“Quillen’s Theorem”

For a multiplicative orientation, the Thom map takes maps of H-spaces in $[BU, \Omega^\infty R^\times] = R^0(BU)$ to maps of ring spectra in $[MU, R] = R^0(MU)$

Theorem (Hirzebruch, Dold, Quillen)

Maps of ring spectra in $R^0(MU)$ are in one-to-one correspondence with elements of $\tilde{R}^2(TU(1))$ that restrict to the unit element of $\tilde{R}^2(S^2)$

When a map of ring spectra $MU \to R$ exists, then:

- *The maps of ring spectra in $R^0(MU)$ are exactly the maps that correspond to maps of H-spaces in $R^0(BU)$ and*
- *Are in one-to-one correspondence with elements of $R^0(BU(1))$ via the map on R^0 induced by $BU(1) \to BU$.*

“Quillen’s Theorem”

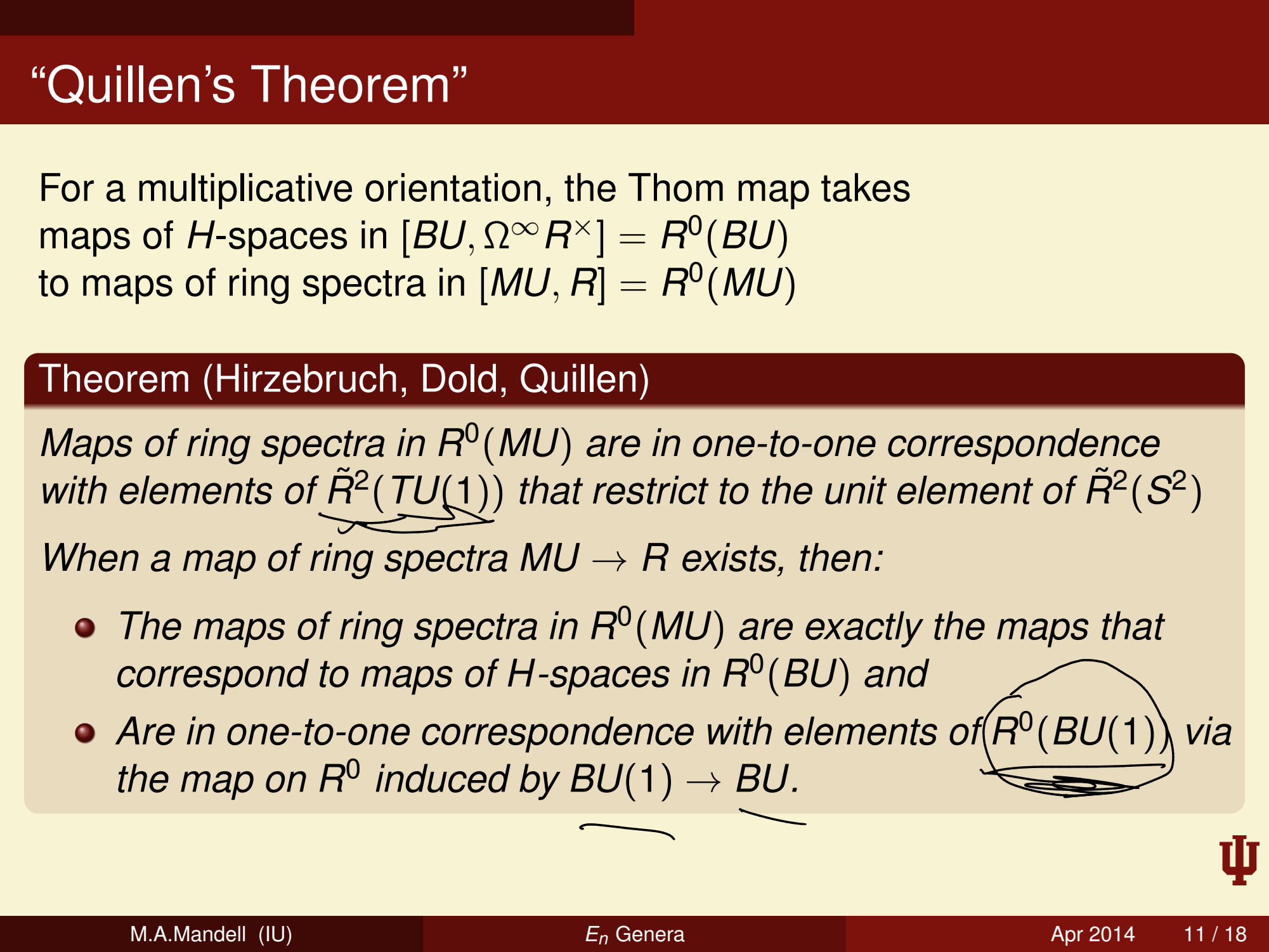
For a multiplicative orientation, the Thom map takes maps of H-spaces in $[BU, \Omega^\infty R^\times] = R^0(BU)$ to maps of ring spectra in $[MU, R] = R^0(MU)$

Theorem (Hirzebruch, Dold, Quillen)

Maps of ring spectra in $R^0(MU)$ are in one-to-one correspondence with elements of $\tilde{R}^2(TU(1))$ that restrict to the unit element of $\tilde{R}^2(S^2)$

When a map of ring spectra $MU \to R$ exists, then:

- *The maps of ring spectra in $R^0(MU)$ are exactly the maps that correspond to maps of H-spaces in $R^0(BU)$ and*
- *Are in one-to-one correspondence with elements of $R^0(BU(1))$ via the map on R^0 induced by $BU(1) \to BU$.*



“Quillen’s Theorem”

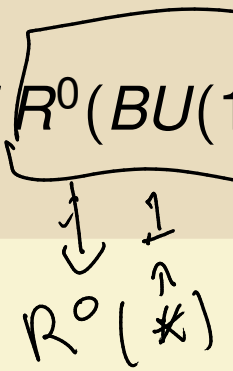
For a multiplicative orientation, the Thom map takes maps of H-spaces in $[BU, \Omega^\infty R^\times] = R^0(BU)$ to maps of ring spectra in $[MU, R] = R^0(MU)$

Theorem (Hirzebruch, Dold, Quillen)

Maps of ring spectra in $R^0(MU)$ are in one-to-one correspondence with elements of $\tilde{R}^2(TU(1))$ that restrict to the unit element of $\tilde{R}^2(S^2)$

When a map of ring spectra $MU \to R$ exists, $\Longleftarrow$ Always holds when R is even

- *The maps of ring spectra in $R^0(MU)$ are exactly the maps that correspond to maps of H-spaces in $R^0(BU)$ and*
- *Are in one-to-one correspondence with elements of $R^0(BU(1))$ via the map on R^0 induced by $BU(1) \to BU$.*



E_n genera

Assume that R is E_n and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \implies MU \to MU \wedge BU_+ \xrightarrow{\sigma\wedge g} R \wedge R \to R$$

Fact:

- Space of E_n ring maps $\Sigma^\infty_+ BU \to R$ isomorphic to space of E_n maps $BU \to \Omega^\infty R^\times$ [May-Quinn-Ray-Tornehave]
- If R is E_n then $R \wedge R \to R$ is E_{n-1} [Dunn]

E_n genera

Assume that R is E_n and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

Fact:

- Space of E_n ring maps $\Sigma^\infty_+ BU \to R$ isomorphic to space of E_n maps $BU \to \Omega^\infty R^\times$ [May-Quinn-Ray-Tornehave]
- If R is E_n then $R \wedge R \to R$ is E_{n-1} [Dunn]

E_n genera

Assume that R is E_n and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma\wedge g} R \wedge R \to R$$

Fact:

- Space of E_n ring maps $\Sigma^\infty_+ BU \to R$ isomorphic to space of E_n maps $BU \to \Omega^\infty R^\times$ [May-Quinn-Ray-Tornehave]
- If R is E_n then $R \wedge R \to R$ is E_{n-1} [Dunn]

E_n genera

Assume that R is E_{n+1} and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \quad \Longrightarrow \quad MU \to MU \wedge BU_+ \xrightarrow{\sigma\wedge g} R \wedge R \to R$$

Fact:

- Space of E_n ring maps $\Sigma^\infty_+ BU \to R$ isomorphic to space of E_n maps $BU \to \Omega^\infty R^\times$ [May-Quinn-Ray-Tornehave]
- If R is E_{n+1} then $R \wedge R \to R$ is E_n [Dunn]

E_n genera

Assume that R is E_{n+1} and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma\wedge g} R \wedge R \to R$$

Fact:

- Space of E_n ring maps $\Sigma^\infty_+ BU \to R$ isomorphic to space of E_n maps $BU \to \Omega^\infty R^\times$ [May-Quinn-Ray-Tornehave]
- If R is E_{n+1} then $R \wedge R \to R$ is E_n [Dunn]

Observation

If R is E_{n+1} and then we have a natural action of the space of E_n maps $\mathcal{E}_n(BU, \Omega^\infty R^\times)$ on the space of E_n ring maps $\mathcal{E}_n(MU, R)$.

If non empty, we get a map $\mathcal{E}_n(BU, \Omega^\infty R^\times) \to \mathcal{E}_n(MU, R)$.

E_n genera

Assume that R is E_{n+1} and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma\wedge g} R \wedge R \to R$$

Fact:

- Space of E_n ring maps $\Sigma^\infty_+ BU \to R$ isomorphic to space of E_n maps $BU \to \Omega^\infty R^\times$ [May-Quinn-Ray-Tornehave]
- If R is E_{n+1} then $R \wedge R \to R$ is E_n [Dunn]

Observation

If R is E_{n+1} and then we have a natural action of the space of E_n maps $\mathcal{E}_n(BU, \Omega^\infty R^\times)$ on the space of E_n ring maps $\mathcal{E}_n(MU, R)$.

If non empty, we get a map $\mathcal{E}_n(BU, \Omega^\infty R^\times) \to \mathcal{E}_n(MU, R)$.

Multiplicative Thom Isomorphism

Assume that R is E_{n+1} and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

induces a map $\mathcal{E}_n(BU, \Omega^\infty R^\times) \simeq \mathcal{E}_n(\Sigma^\infty_+ BU, R) \to \mathcal{E}_n(MU, R)$

Theorem

This map is a weak equivalence.

- Suffices to consider the case when R is connective
- Look at Postnikov tower of R

Multiplicative Thom Isomorphism

Assume that R is E_{n+1} and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

induces a map $\mathcal{E}_n(BU, \Omega^\infty R^\times) \simeq \mathcal{E}_n(\Sigma^\infty_+ BU, R) \to \mathcal{E}_n(MU, R)$

Theorem

This map is a weak equivalence.

- Suffices to consider the case when R is connective
- Look at Postnikov tower of R

Multiplicative Thom Isomorphism

Assume that R is E_{n+1} and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

induces a map $\mathcal{E}_n(BU, \Omega^\infty R^\times) \simeq \mathcal{E}_n(\Sigma^\infty_+ BU, R) \to \mathcal{E}_n(MU, R)$

Theorem

This map is a weak equivalence.

- Suffices to consider the case when R is connective
- Look at Postnikov tower of R

Multiplicative Thom Isomorphism

Assume that R is E_{n+1} and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

induces a map $\mathcal{E}_n(BU, \Omega^\infty R^\times) \simeq \mathcal{E}_n(\Sigma^\infty_+ BU, R) \to \mathcal{E}_n(MU, R)$

Theorem

This map is a weak equivalence.

- Suffices to consider the case when R is connective
- Look at Postnikov tower of R

Postnikov Towers of E_n ring spectra

Let R be a connective E_n ring spectrum and let $\underline{Z} = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2\rangle \to R\langle 1\rangle \to R\langle 0\rangle = HZ$$

in the category of E_n ring spectra.

Postnikov Towers of E_n ring spectra

Let R be a connective E_n ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2\rangle \to R\langle 1\rangle \to R\langle 0\rangle = HZ$$

in the category of E_n ring spectra.

Postnikov Towers of E_n ring spectra

Let R be a connective E_n ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2\rangle \to R\langle 1\rangle \to R\langle 0\rangle = HZ$$

in the category of E_n ring spectra.

Postnikov Towers of E_n ring spectra

Let R be a connective E_n ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2\rangle \to R\langle 1\rangle \to R\langle 0\rangle = HZ$$

in the category of E_n ring spectra.

Postnikov Towers of E_n ring spectra

Let R be a connective E_n ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2\rangle \to R\langle 1\rangle \to R\langle 0\rangle = HZ$$

in the category of E_n ring spectra.

Postnikov Towers of E_n ring spectra

Let R be a connective E_n ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2 \rangle \to R\langle 1 \rangle \to R\langle 0 \rangle = HZ$$

in the category of E_n ring spectra.

Postnikov Towers of E_n ring spectra

Let R be a connective E_n ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2 \rangle \to R\langle 1 \rangle \to R\langle 0 \rangle = HZ$$

in the category of E_n ring spectra.

Postnikov Towers of E_n ring spectra

Let R be a connective E_n ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2\rangle \to R\langle 1\rangle \to R\langle 0\rangle = HZ$$

in the category of E_n ring spectra.

Magic Fact (Kriz)

This is a tower of principal fibrations of E_n ring spectra

$$\begin{array}{ccc} R\langle j+1\rangle & \longrightarrow & HZ \\ \downarrow & & \downarrow \\ R\langle j\rangle & \xrightarrow[k_{j+1}]{} & HZ \vee \Sigma^{j+2} H\pi_{j+1} R \end{array}$$

Postnikov Towers of E_n ring spectra

Let R be a connective E_n ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2 \rangle \to R\langle 1 \rangle \to R\langle 0 \rangle = HZ$$

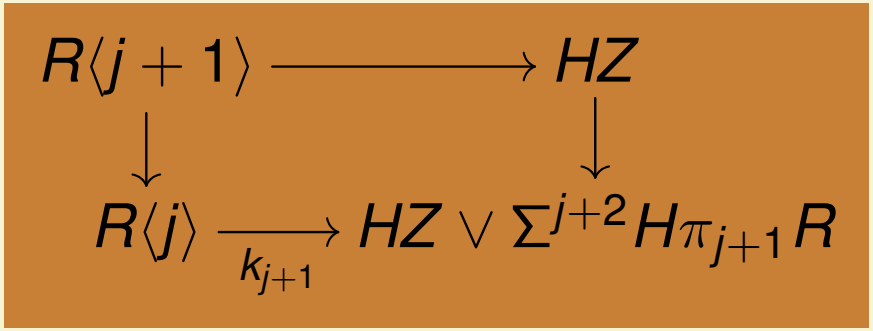
in the category of E_n ring spectra.

Magic Fact (Kriz)

This is a tower of principal fibrations of E_n ring spectra

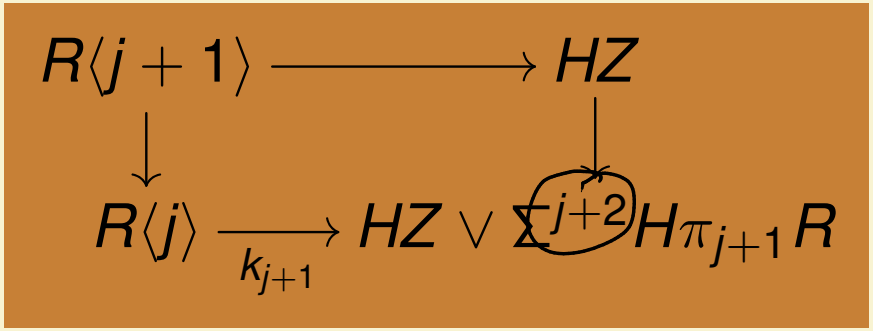
$$\begin{array}{ccc} R\langle j+1 \rangle & \longrightarrow & HZ \\ \downarrow & & \downarrow \\ R\langle j \rangle & \xrightarrow[k_{j+1}]{} & HZ \vee \Sigma^{j+2} H\pi_{j+1} R \end{array}$$

Topological Quillen Cohomology



Topological Quillen Cohomology

$$H^*_{\mathcal{E}_n}(A; M) := \pi_0 \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^* HM)$$



Obstruction theory

- Obstruction in $H^{j+2}(A; \pi_{j+1} R)$ to lifting an E_n ring map $A \to R\langle j\rangle$ to an E_n ring map $A \to R\langle j+1\rangle$
- The space of lifts is either empty or is a "free orbit" on the grouplike topological monoid

$$\mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+1} H\pi_{j+1} R) \simeq \Omega \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+2} H\pi_{j+1} R)$$

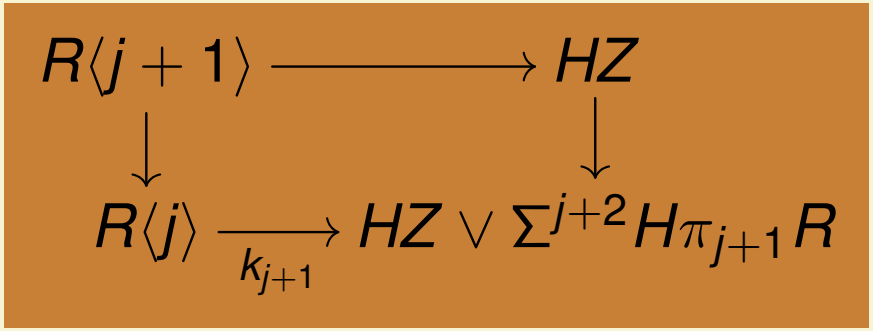
Atiyah-Hirzebruch Spectral Sequence

For E_n ring spectra A, R (with mild hypotheses on A), there is a natural "obstructed" spectral sequence

$$E^2_{p,q} = H^p_{\mathcal{E}_n}(A; \pi_q R) \implies \pi_{q-p} \mathcal{E}_n(A, R).$$

Topological Quillen Cohomology

$$H^*_{\mathcal{E}_n}(A; M) := \pi_0 \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^* HM)$$



Obstruction theory

- Obstruction in $H^{j+2}(A; \pi_{j+1} R)$ to lifting an E_n ring map $A \to R\langle j\rangle$ to an E_n ring map $A \to R\langle j+1\rangle$
- The space of lifts is either empty or is a "free orbit" on the grouplike topological monoid

$$\mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+1} H\pi_{j+1} R) \simeq \Omega \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+2} H\pi_{j+1} R)$$

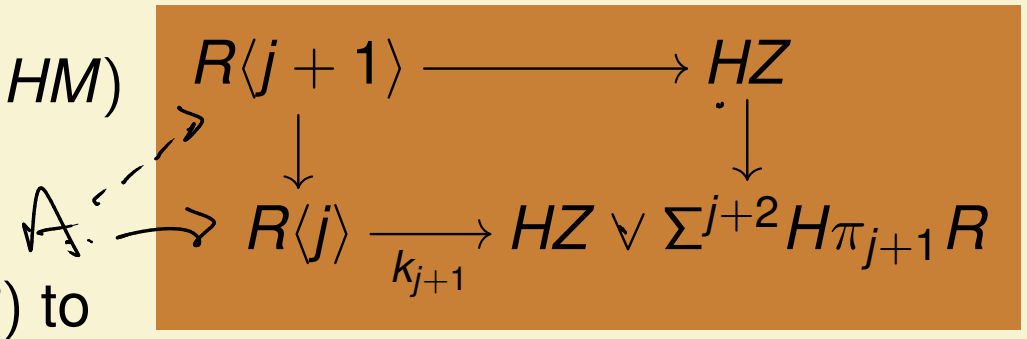
Atiyah-Hirzebruch Spectral Sequence

For E_n ring spectra A, R (with mild hypotheses on A), there is a natural "obstructed" spectral sequence

$$E^2_{p,q} = H^p_{\mathcal{E}_n}(A; \pi_q R) \implies \pi_{q-p} \mathcal{E}_n(A, R).$$

Topological Quillen Cohomology

$$H^*_{\mathcal{E}_n}(A; M) := \pi_0 \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^* HM)$$



Obstruction theory

- Obstruction in $H^{j+2}_{\mathcal{E}_n}(A; \pi_{j+1} R)$ to lifting an E_n ring map $A \to R\langle j\rangle$ to an E_n ring map $A \to R\langle j+1\rangle$
- The space of lifts is either empty or is a "free orbit" on the grouplike topological monoid

$$\mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+1} H\pi_{j+1} R) \simeq \Omega \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+2} H\pi_{j+1} R)$$

Atiyah-Hirzebruch Spectral Sequence

For E_n ring spectra A, R (with mild hypotheses on A), there is a natural "obstructed" spectral sequence

$$E^2_{p,q} = H^p_{\mathcal{E}_n}(A; \pi_q R) \implies \pi_{q-p} \mathcal{E}_n(A, R).$$

Topological Quillen Cohomology

$$H^*_{\mathcal{E}_n}(A; M) := \pi_0 \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^* HM)$$

$$\begin{array}{ccc} R\langle j+1\rangle & \longrightarrow & HZ \\ \downarrow & & \downarrow \\ R\langle j\rangle & \xrightarrow[k_{j+1}]{} & HZ \vee \Sigma^{j+2} H\pi_{j+1} R \end{array}$$

Obstruction theory

- Obstruction in $H^{j+2}(A; \pi_{j+1} R)$ to lifting an E_n ring map $A \to R\langle j\rangle$ to an E_n ring map $A \to R\langle j+1\rangle$
- The space of lifts is either empty or is a "free orbit" on the grouplike topological monoid

$$\mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+1} H\pi_{j+1} R) \simeq \Omega \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+2} H\pi_{j+1} R)$$

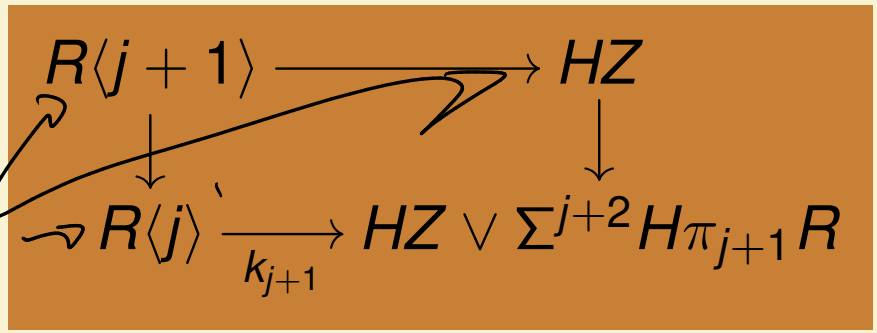
Atiyah-Hirzebruch Spectral Sequence

For E_n ring spectra A, R (with mild hypotheses on A), there is a natural "obstructed" spectral sequence

$$E^2_{p,q} = H^p_{\mathcal{E}_n}(A; \pi_q R) \implies \pi_{q-p} \mathcal{E}_n(A, R).$$

Topological Quillen Cohomology

$$H^*_{\mathcal{E}_n}(A; M) := \pi_0 \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^* HM)$$



Obstruction theory

- Obstruction in $H^{j+2}(A; \pi_{j+1} R)$ to lifting an E_n ring map $A \to R\langle j\rangle$ to an E_n ring map $A \to R\langle j+1\rangle$
- The space of lifts is either empty or is a "free orbit" on the grouplike topological monoid

$$\mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+1} H\pi_{j+1} R) \simeq \Omega \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+2} H\pi_{j+1} R)$$

Atiyah-Hirzebruch Spectral Sequence

For E_n ring spectra A, R (with mild hypotheses on A), there is a natural "obstructed" spectral sequence

$$E^2_{p,q} = H^p_{\mathcal{E}_n}(A; \pi_q R) \implies \pi_{q-p} \mathcal{E}_n(A, R).$$

Topological Quillen Cohomology

$$H^*_{\mathcal{E}_n}(A; M) := \pi_0 \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^* HM)$$

$$\begin{array}{ccc} R\langle j+1\rangle & \longrightarrow & HZ \\ \downarrow & & \downarrow \\ R\langle j\rangle & \xrightarrow[k_{j+1}]{} & HZ \vee \Sigma^{j+2} H\pi_{j+1} R \end{array}$$

Obstruction theory

- Obstruction in $H^{j+2}(A; \pi_{j+1} R)$ to lifting an E_n ring map $A \to R\langle j\rangle$ to an E_n ring map $A \to R\langle j+1\rangle$
- The space of lifts is either empty or is a "free orbit" on the ~~grouplike topological monoid~~

$$\mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+1} H\pi_{j+1} R) \simeq \Omega \mathcal{E}_{n/HZ}(A, HZ \vee \Sigma^{j+2} H\pi_{j+1} R)$$

Atiyah-Hirzebruch Spectral Sequence

For E_n ring spectra A, R (with mild hypotheses on A), there is a natural "obstructed" spectral sequence

$$E^2_{p,q} = H^p_{\mathcal{E}_n}(A; \pi_q R) \Longrightarrow \pi_{q-p} \mathcal{E}_n(A, R).$$

Multiplicative Thom Isomorphism

Thom isomorphism: $HZ \wedge MU \xrightarrow{\simeq} HZ \wedge BU_+$ as E_n HZ-algebras.

$$HZ \wedge MU \to HZ \wedge MU \wedge BU_+ \to HZ \wedge HZ \wedge BU_+ \to HZ \wedge BU_+$$

$$\Longrightarrow \quad H^*_{\mathcal{E}_n}(\Sigma^\infty_+ BU; -) \xrightarrow{\simeq} H^*_{\mathcal{E}_n}(MU; -)$$

Consequence

For R an E_{n+1} ring spectrum and $\sigma : MU \to R$ an E_n ring map, the Thom map induces an isomorphism on E^2-terms

$$H^p_{\mathcal{E}_n}(\Sigma^\infty_+ BU; \pi_q R) \xrightarrow{\cong} H^p_{\mathcal{E}_n}(MU; \pi_q R)$$

and an isomorphism

$$\pi_* \mathcal{E}_n(\Sigma^\infty_+ BU, R) \xrightarrow{\cong} \pi_* \mathcal{E}_n(MU, R)$$

Nothing special about BU/MU here; works for any E_n Thom spectrum.

Multiplicative Thom Isomorphism

Thom isomorphism: $HZ \wedge MU \xrightarrow{\simeq} HZ \wedge BU_+$ as E_n HZ-algebras.

$$HZ \wedge MU \to HZ \wedge MU \wedge BU_+ \to HZ \wedge HZ \wedge BU_+ \to HZ \wedge BU_+$$

$$\implies \qquad H^*_{\mathcal{E}_n}(\Sigma^\infty_+ BU; -) \xrightarrow{\simeq} H^*_{\mathcal{E}_n}(MU; -)$$

Consequence

For R an E_{n+1} ring spectrum and $\sigma : MU \to R$ an E_n ring map, the Thom map induces an isomorphism on E^2-terms

$$H^p_{\mathcal{E}_n}(\Sigma^\infty_+ BU; \pi_q R) \xrightarrow{\cong} H^p_{\mathcal{E}_n}(MU; \pi_q R)$$

and an isomorphism

$$\pi_* \mathcal{E}_n(\Sigma^\infty_+ BU, R) \xrightarrow{\cong} \pi_* \mathcal{E}_n(MU, R)$$

Nothing special about BU/MU here; works for any E_n Thom spectrum.

Multiplicative Thom Isomorphism

Thom isomorphism: $HZ \wedge MU \xrightarrow{\simeq} HZ \wedge BU_+$ as E_n HZ-algebras.

$$HZ \wedge MU \to HZ \wedge MU \wedge BU_+ \to HZ \wedge HZ \wedge BU_+ \to HZ \wedge BU_+$$

$$\implies \qquad H^*_{\mathcal{E}_n}(\Sigma^\infty_+ BU; -) \xrightarrow{\simeq} H^*_{\mathcal{E}_n}(MU; -)$$

Consequence

For R an E_{n+1} ring spectrum and $\sigma : MU \to R$ an E_n ring map, the Thom map induces an isomorphism on E^2-terms

$$H^p_{\mathcal{E}_n}(\Sigma^\infty_+ BU; \pi_q R) \xrightarrow{\cong} H^p_{\mathcal{E}_n}(MU; \pi_q R)$$

and an isomorphism

$$\pi_* \mathcal{E}_n(\Sigma^\infty_+ BU, R) \xrightarrow{\cong} \pi_* \mathcal{E}_n(MU, R)$$

Nothing special about BU/MU here; works for any E_n Thom spectrum.

Multiplicative Thom Isomorphism

Thom isomorphism: $HZ \wedge MU \xrightarrow{\simeq} HZ \wedge BU_+$ as E_n HZ-algebras.

$$HZ \wedge MU \to HZ \wedge MU \wedge BU_+ \to HZ \wedge HZ \wedge BU_+ \to HZ \wedge BU_+$$

$$\Longrightarrow \qquad H^*_{\mathcal{E}_n}(\Sigma^\infty_+ BU; -) \xrightarrow{\simeq} H^*_{\mathcal{E}_n}(MU; -)$$

Consequence

For R an E_{n+1} ring spectrum and $\sigma : MU \to R$ an E_n ring map, the Thom map induces an isomorphism on E^2-terms

$$H^p_{\mathcal{E}_n}(\Sigma^\infty_+ BU; \pi_q R) \xrightarrow{\cong} H^p_{\mathcal{E}_n}(MU; \pi_q R)$$

and an isomorphism

$$\pi_* \mathcal{E}_n(\Sigma^\infty_+ BU, R) \xrightarrow{\cong} \pi_* \mathcal{E}_n(MU, R)$$

Nothing special about BU/MU here; works for any E_n Thom spectrum.

Computation

$$\begin{aligned}\mathcal{E}_n(\Sigma^\infty_+ BU, R) &\simeq \mathcal{E}_n(BU, \Omega^\infty R^\times) \\ &= \mathcal{E}_n(BU, (\Omega^\infty R^\times)_1) \\ &\simeq \mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)\end{aligned}$$

Compute using "Atiyah-Hirzebruch spectral sequence"

$$H^p(B^n BU; R_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega^\infty R)_1)) \implies \pi_{q+n-p}\mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)$$

For $n = 2$,

$$H^*(B^2 BU) = H^*(BSU) = \mathbb{Z}[c_2, c_3, \ldots]$$

and

$$H^*(BSU) \to H^*(\Sigma^2 BU(1))$$

is surjective.

Computation

$$\begin{aligned}\mathcal{E}_n(\Sigma^\infty_+ BU, R) &\simeq \mathcal{E}_n(BU, \Omega^\infty R^\times) \\ &= \mathcal{E}_n(BU, (\Omega^\infty R^\times)_1) \\ &\simeq \mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)\end{aligned}$$

Compute using "Atiyah-Hirzebruch spectral sequence"

$$H^p(B^n BU; R_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega^\infty R)_1)) \implies \pi_{q+n-p}\mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)$$

For $n = 2$,

$$H^*(B^2 BU) = H^*(BSU) = \mathbb{Z}[c_2, c_3, \ldots]$$

and

$$H^*(BSU) \to H^*(\Sigma^2 BU(1))$$

is surjective.

Computation

$$\begin{aligned}\mathcal{E}_n(\Sigma^\infty_+ BU, R) &\simeq \mathcal{E}_n(BU, \Omega^\infty R^\times) \\ &= \mathcal{E}_n(BU, (\Omega^\infty R^\times)_1) \\ &\simeq \mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)\end{aligned}$$

Compute using "Atiyah-Hirzebruch spectral sequence"

$$H^p(B^n BU; R_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega^\infty R)_1)) \implies \pi_{q+n-p}\mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)$$

For $n = 2$,

$$H^*(B^2 BU) = H^*(BSU) = \mathbb{Z}[c_2, c_3, \ldots]$$

and

$$H^*(BSU) \to H^*(\Sigma^2 BU(1))$$

is surjective.

Computation

$$\begin{aligned}\mathcal{E}_n(\Sigma^\infty_+ BU, R) &\simeq \mathcal{E}_n(BU, \Omega^\infty R^\times) \\ &= \mathcal{E}_n(BU, (\Omega^\infty R^\times)_1) \\ &\simeq \mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)\end{aligned}$$

Compute using "Atiyah-Hirzebruch spectral sequence"

$$H^p(B^n BU; R_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega^\infty R)_1)) \implies \pi_{q+n-p}\mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)$$

For $n = 2$,

$$H^*(B^2 BU) = H^*(BSU) = \mathbb{Z}[c_2, c_3, \ldots]$$

and

$$H^*(BSU) \to H^*(\Sigma^2 BU(1))$$

is surjective.

Computation

$$\begin{aligned}\mathcal{E}_n(\Sigma^\infty_+ BU, R) &\simeq \mathcal{E}_n(BU, \Omega^\infty R^\times) \\ &= \mathcal{E}_n(BU, (\Omega^\infty R^\times)_1) \\ &\simeq \mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)\end{aligned}$$

Compute using "Atiyah-Hirzebruch spectral sequence"

$$H^p(B^n BU; R_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega^\infty R)_1)) \implies \pi_{q+n-p}\mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)$$

For $n = 2$,

$$H^*(B^2 BU) = H^*(BSU) = \mathbb{Z}[c_2, c_3, \ldots]$$

and

$$H^*(BSU) \to H^*(\Sigma^2 BU(1))$$

is surjective.

Computation

$$\begin{aligned}\mathcal{E}_n(\Sigma^\infty_+ BU, R) &\simeq \mathcal{E}_n(BU, \Omega^\infty R^\times) \\ &= \mathcal{E}_n(BU, (\Omega^\infty R^\times)_1) \\ &\simeq \mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)\end{aligned}$$

Compute using "Atiyah-Hirzebruch spectral sequence"

$$H^p(B^n BU; R_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega^\infty R)_1)) \implies \pi_{q+n-p}\mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)$$

For $n = 2$,

$$H^*(B^2 BU) = H^*(BSU) = \mathbb{Z}[c_2, c_3, \ldots]$$

and

$$H^*(BSU) \to H^*(\Sigma^2 BU(1))$$

is surjective.

Computation

$$\begin{aligned}\mathcal{E}_n(\Sigma^\infty_+ BU, R) &\simeq \mathcal{E}_n(BU, \Omega^\infty R^\times) \\ &= \mathcal{E}_n(BU, (\Omega^\infty R^\times)_1) \\ &\simeq \mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)\end{aligned}$$

Compute using "Atiyah-Hirzebruch spectral sequence"

$$H^p(B^n BU; R_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega^\infty R)_1)) \implies \pi_{q+n-p}\mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1)$$

For $n = 2$,

$$H^*(B^2 BU) = H^*(BSU) = \mathbb{Z}[c_2, c_3, \ldots]$$

and

$$H^*(BSU) \to H^*(\Sigma^2 BU(1))$$

is surjective.

Main Result

Theorem

If R is an even E_3 ring spectrum, then $\mathcal{E}_2(MU, R) \simeq \mathcal{E}_2(BU, \Omega^\infty R^\times)$ and $\pi_{-*}\mathcal{E}_2(BU, \Omega^\infty R^\times) \to R^*(BU(1))$ is surjective. Thus, every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

What about for R just E_2?

$$H^*_{\mathcal{E}_2}(MU; \pi) \cong H^*_{\mathcal{E}_2}(\Sigma^\infty_+ BU; \pi) \cong H^{*+2}(B^n BU; \pi)$$
$$= H^{*+2}(BSU; \pi) \twoheadrightarrow H^*(BU(1); \pi)$$

Careful argument with "Atiyah-Hirzebruch spectral sequence"

Theorem

If R is an even E_2 ring spectrum then every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

Main Result

Theorem

If R is an even E_3 ring spectrum, then $\mathcal{E}_2(MU, R) \simeq \mathcal{E}_2(BU, \Omega^\infty R^\times)$ and $\pi_{-*}\mathcal{E}_2(BU, \Omega^\infty R^\times) \to R^*(BU(1))$ is surjective. Thus, every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

What about for R just E_2?

$$H^*_{\mathcal{E}_2}(MU;\pi) \cong H^*_{\mathcal{E}_2}(\Sigma^\infty_+ BU;\pi) \cong H^{*+2}(B^n BU;\pi)$$
$$= H^{*+2}(BSU;\pi) \twoheadrightarrow H^*(BU(1);\pi)$$

Careful argument with "Atiyah-Hirzebruch spectral sequence"

Theorem

If R is an even E_2 ring spectrum then every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

Main Result

Theorem

If R is an even E_3 ring spectrum, then $\mathcal{E}_2(MU, R) \simeq \mathcal{E}_2(BU, \Omega^\infty R^\times)$ and $\pi_{-*}\mathcal{E}_2(BU, \Omega^\infty R^\times) \to R^*(BU(1))$ is surjective. Thus, every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

What about for R just E_2?

$$H^*_{\mathcal{E}_2}(MU; \pi) \cong H^*_{\mathcal{E}_2}(\Sigma^\infty_+ BU; \pi) \cong H^{*+2}(B^n BU; \pi)$$

$$= H^{*+2}(BSU; \pi) \twoheadrightarrow H^*(BU(1); \pi)$$

Careful argument with "Atiyah-Hirzebruch spectral sequence"

Theorem

If R is an even E_2 ring spectrum then every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

Main Result

Theorem

If R is an even E_3 ring spectrum, then $\mathcal{E}_2(MU, R) \simeq \mathcal{E}_2(BU, \Omega^\infty R^\times)$ and $\pi_{-*}\mathcal{E}_2(BU, \Omega^\infty R^\times) \to R^*(BU(1))$ is surjective. Thus, every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

What about for R just E_2?

$$H^*_{\mathcal{E}_2}(MU; \pi) \cong H^*_{\mathcal{E}_2}(\Sigma^\infty_+ BU; \pi) \cong H^{*+2}(B^n BU; \pi)$$

$$= H^{*+2}(BSU; \pi) \twoheadrightarrow H^*(BU(1); \pi)$$

Careful argument with "Atiyah-Hirzebruch spectral sequence"

Theorem

If R is an even E_2 ring spectrum then every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

Main Result

Theorem

If R is an even E_3 ring spectrum, then $\mathcal{E}_2(MU, R) \simeq \mathcal{E}_2(BU, \Omega^\infty R^\times)$ and $\pi_{-*}\mathcal{E}_2(BU, \Omega^\infty R^\times) \to R^*(BU(1))$ is surjective. Thus, every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

What about for R just E_2?

$$H^*_{\mathcal{E}_2}(MU; \pi) \cong H^*_{\mathcal{E}_2}(\Sigma^\infty_+ BU; \pi) \cong H^{*+2}(B^n BU; \pi)$$
$$= H^{*+2}(BSU; \pi) \twoheadrightarrow H^*(BU(1); \pi)$$

Careful argument with "Atiyah-Hirzebruch spectral sequence"

Theorem

If R is an even E_2 ring spectrum then every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

Main Result

Theorem

If R is an even E_3 ring spectrum, then $\mathcal{E}_2(MU, R) \simeq \mathcal{E}_2(BU, \Omega^\infty R^\times)$ and $\pi_{-*}\mathcal{E}_2(BU, \Omega^\infty R^\times) \to R^*(BU(1))$ is surjective. Thus, every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

What about for R just E_2?

$$H^*_{\mathcal{E}_2}(MU; \pi) \cong H^*_{\mathcal{E}_2}(\Sigma^\infty_+ BU; \pi) \cong H^{*+2}(B^n BU; \pi)$$
$$= H^{*+2}(BSU; \pi) \twoheadrightarrow H^*(BU(1); \pi)$$

Careful argument with "Atiyah-Hirzebruch spectral sequence"

Theorem

If R is an even E_2 ring spectrum then every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.