MOTIVES VERSUS NONCOMMUTATIVE MOTIVES

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We'll start with motives, discuss a commutative problem, do a non-commutative walk (through non-commutative motives), and come up with a partial solution to that problem.

1. Pure Motives

Let k be a base field and denote by SmProj(k) the category of smooth projective k-schemes.

Chow(k)$_{\mathbb{Q}}$ is the category of Chow motives. It's the idempotent completion of the category with objects (X, r) where $X \in SmProj(k)$, $r \in \mathbb{Z}$, morphisms $Hom((X, r), (Y, r')) = Z_{rat}^{dimX-r+r'}(X \times Y)_{\mathbb{Q}}$ where Z stands for algebraic cycles of fixed codimension. The product is $(X, r) \otimes (Y, r') = (X \times Y, r + r')$.

There's a functor $SmProj(k)^{op} \to Chow(k)_{\mathbb{Q}}$ given by $X \to (X, 0)$ and a map $f : X \to Y$ to the graph $\Gamma(f)$.

Let $(C, \otimes, 1)$ be a $\mathbb{Q}$-linear additive rigid symmetric monoidal category. Define $\otimes_{Nil}(a, b) = \{f \in \mathrm{Hom}_C(a, b) \mid f^{\otimes n} = 0, n >> 0\}$ and $N(a, b) = \{f : a \to b \mid \forall g : b \to a, trace(g \circ f) = 0\}$.

Grothendieck already knew this. He wanted $\otimes$-nilpotent motives and numerical motives, both of which can be built from the category $Chow(k)_{\mathbb{Q}}$ by quotienting out by the sets above (respectively) and taking idempotent completion of the result. Call the first $Voev(k)_{\mathbb{Q}}$ and call the second one $Num(k)_{\mathbb{Q}}$.

Janssen proved $Num(k)_{\mathbb{Q}}$ is an abelian semi-simple category.

There is a functor $SmProj(k)^{op} \to Chow(k)_{\mathbb{Q}} \to Voev(k)_{\mathbb{Q}} \to Num(k)_{\mathbb{Q}}$.

2. Non-Commutative Pure Motives

A DG-category is one enriched in complexes of k-vector spaces. A dg k-algebra A gives rise to a DG-category on one object. Any k-scheme X gives $Perf_{dg}(X)$, a DG-category.

A dg-category $\mathcal{A}$ is called smooth if $\mathcal{A} \in D_c(\mathcal{A}^{op} \otimes \mathcal{A})$. It's called proper if the sum over all $\dim H^i(\mathcal{A}(x, y))$ is finite.

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Fact: X is a smooth and proper scheme iff $Perf_{dg}(X)$ is smooth and proper.

Let SmProp(k) be the category of smooth and proper dg categories.

We can form a category of non-commutative Chow motives $NChow(k)_{\mathbb{Q}}$ as the idempotent completion of a category whose objects are SmProp(k) and whose morphisms Hom($\mathcal{A}, \mathcal{B}$) is defined to be the Grothendieck group $K_0(\mathcal{A}^{op} \otimes \mathcal{B})_{\mathbb{Q}}$. There is again a monoidal product as in the commutative case.

There is a functor U:SmProp(k)$\rightarrow$NChow(k)$_{\mathbb{Q}}$ which is the identity on objects and takes a map $\mathcal{A} \rightarrow \mathcal{B}$ to $[_F\mathcal{B}]$.

As before we can define noncommutative nilpotent motives $NVoev(k)_{\mathbb{Q}}$ and non-commutative numerical motives by quotienting by the same sets as before and taking idempotent completion.

Theorem 2.1. *$Chow(k) = 0$ implies $NNum(k)_{\mathbb{Q}}$ is abelian and semi-simple.*

There is a chain of functors SmProp(k)$\rightarrow$NChow$(k)_{\mathbb{Q}} \rightarrow$ NVoev$(k)_{\mathbb{Q}} \rightarrow$ NNum$(k)_{\mathbb{Q}}$

3. Orbit categories

Let $(C, \otimes, 1)$ be a $\mathbb{Q}$-linear additive symmetric monoidal category. Suppose O is a $\otimes$-invertible object of C. So $- \otimes O$ is an automorphism.

The orbit category is $C/{-\otimes O}$ with objects of C and $\mathrm{Hom}(a, b) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_C(a, b \otimes O^{\otimes n})$

There is a functor $\pi : C \rightarrow C/-\otimes O$ and the natural isomorphism $\pi \circ (- \otimes O) \Rightarrow \pi$ is 2-universal.

Example: $M(\mathbb{P}^1)$ is the Chow motive of the projective line. It decomposes into $M(Spec(k)) \oplus (Spec(k), -1)$, where the second piece is the Lefschetz motive. There is also the Tate motive $\mathbb{Q}(1)$ and we are interested in the orbit category $Chow(k)_{\mathbb{Q}}/-\otimes\mathbb{Q}(1)$.

4. Bridges

We have seen that from SmProj$(k)^{op}$ one can get all the way to Num$(k)_{\otimes}/-\otimes\mathbb{Q}(1)$. In fact you can do this in several ways e.g. passing to the orbit category at any of the levels in the chain of functors we finished Section 1 with.

We can similarly pass from SmProp(k) to NNum(k)$_{\mathbb{Q}}$ on the non-commutative side.

There is a map $X \rightarrow Perf_{dg}X$ from SmProj to SmProp.

Theorem: There are $\mathbb{Q}$-linear fully faithful $\otimes$-functors R, R_{nil}, R_N which go from commutative to non-commutative at each level, e.g. $R : Chow(k)_{\mathbb{Q}}/-\otimes\mathbb{Q}(1) \rightarrow$

$NChow(k)_{\mathbb{Q}}$ and the others are similarly going from the orbit category on the commutative side to the corresponding category on the non-commutative side. These functors make the corresponding diagram of categories commute:

$$\begin{array}{ccc}
SmProj(k)^{op} & \longrightarrow & SmProp(k) \\
\downarrow & & \downarrow \\
Chow(k)_{\mathbb{Q}}/-\otimes\mathbb{Q}(1) & \longrightarrow & NChow(k)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
Voev(k)_{\mathbb{Q}}/-\otimes\mathbb{Q}(1) & \longrightarrow & NVoev(k)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
Num(k)_{\mathbb{Q}}/-\otimes\mathbb{Q}(1) & \longrightarrow & NNum(k)_{\mathbb{Q}}
\end{array}$$

The following is joint with Morcolli.

Theorem 4.1. *Suppose Perf(X) is $\langle \epsilon_1, \dots, \epsilon_n \rangle$ where ϵ_i are objects satisfying the property that they generate all objects in Perf(X) and that hom's between ϵ_i and ϵ_j are k if $i = j$ and 0 otherwise.*

Then $M(X)_{\mathbb{Q}} \simeq \mathbb{L}^{\otimes \ell_1} \oplus \cdots \oplus \mathbb{L}^{\otimes \ell_n}$ where $\ell_1, \dots, \ell_n \in \{0, \dots, dim(X)\}$.

The proof is just the commutative diagram above. It connects something geometric with something categorical.

5. Problem 2: Voevodsky's Nilpotence Conjecture

This conjecture was made in 1995: Let X be a smooth projective scheme. Then $Z^*_{nil}(X)_{\mathbb{Q}} = Z^*_{num}(X)_{\mathbb{Q}}$, i.e. cycles up to nilpotent rational equivalence are the same as cycles up to numerical rational equivalence.

This conjecture would imply one of Grothendieck's standard conjectures.

It's known for curves, surfaces, and abelian 3-folds (char(k) = 0).

We can extend this conjecture to be a statement about $\mathcal{A} \in SmProp(k)$ rather than about $X \in SmProj(k)$. First, notation: $K_0(\mathcal{A})_{\mathbb{Q}}/\sim \otimes nil = \mathrm{Hom}_{NVoev(k)_{\mathbb{Q}}}(U(k), U(\mathcal{A})$ and $K_0(\mathcal{A})_{\mathbb{Q}}/\sim num = \mathrm{Hom}_{NNum(k)_{\mathbb{Q}}}(U(k), U(\mathcal{A})$.

Voevodsky's non-commutative conjecture then says $K_0(\mathcal{A})_{\mathbb{Q}}/\sim \otimes nil = K_0(\mathcal{A})_{\mathbb{Q}}/\sim num$.

Theorem: Voevodsky's conjecture is true iff the non-commutative version is true. There are also examples where it's known. This is joint with Bernardara.

Examples:

(1) Quadric fibrations $Q \to s$ of relative dimension n (at least, when n is even and dim(s)≤ 2).

(2) Intersections of quadrics

(3) Linear sections of grassmannians

(4) Moishezon manifolds

6. Problem 3: Panajape's Conjecture

Let $X \subset \mathbb{P}^n$ be a complete intersection of multi-degree $(d_1, \ldots, d_r)$ and let $\alpha = [m - \sum_{i>1} d_i/d_1]$

Conjecture PS(X): When $i < \alpha$ the Chow ring is quite simple. In particular, $Z_{rat}i(X)_{\mathbb{Q}} = \mathbb{Q}$.

There are some partial results on this classically, by Esnault, Levine, Viehweg.

Attacking this problem non-commutative requires non-commutative Jacobians. So we'll work over $k = \mathbb{C}$ and $char(X) = d$. Consider non-commutative deRham cohomology.

$$NH^{2i+1}_{dR}(X) = \Sigma_{C,M(C)\to M(X)\otimes\mathbb{Q}(i)} Im(H^1_{dR}(C) \to H^{2i+1}_{dR}(X))$$

The classical intersection pairing $\langle -, - \rangle$ restricts to give $NH^{2d-2i-1}_{dR}(X) \times NH^{2i+1}_{dR}(X) \to \mathbb{C}$.

Theorem with Marcolli: There is a $\mathbb{Q}$-linear additive Jacobian functor $J(-) : NChow(k)_{\mathbb{Q}} \to Ab(k)_{\mathbb{Q}}$.

Furthermore, if the pairing is non-degenerate then $J(Perf_{dg}(X)) \simeq \prod_{i=0}^{d-1} J^n_i(X)$.

Corollary 6.1. *With no condition on the pairing, if X is a curve C then $J(Perf_{dg}(C)) \simeq J(C)$, and if X is a surface S then $J(Perf_{dg}(S)) \simeq Pic^0(S) \times Ab(S)$.*

For a quadric fibration $Q \to \mathbb{P}^2$, $J(Perf_{dg}(\mathbb{P}^2, C_0)$ splits as a product of $J^a_i(Q)$.

Theorem (with Bernardara): The conjecture PS(X) holds when X is a complete intersection of two quadrics or of three odd dimensional quadrics.

This improves on the proof of Esnault, Levine, Viehweg by getting it to work in ambient spaces of arbitrary dimension (rather than just low dimension). However, this result only applies to a restricted class of X.