Huang- Hypersurfaces

Fig 1

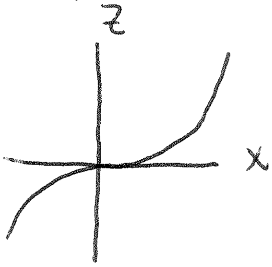
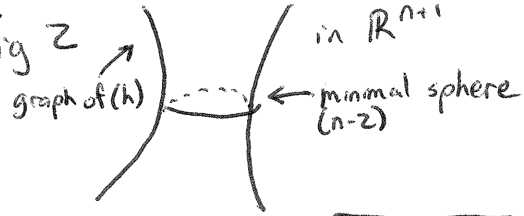


Fig 2



$$h(r) = \begin{cases} \sqrt{8m(|x|-2m)} & n=3 \\ \sqrt{2m}\ \ln\left(|x| + \sqrt{|x|^2 - 2m}\right) & n=4 \\ O\left(|x|^{2-\frac{n}{2}}\right) & n>5 \end{cases}$$

Fig 3

Fig 4

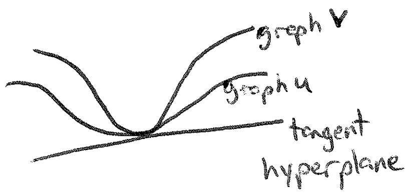


Fig 5

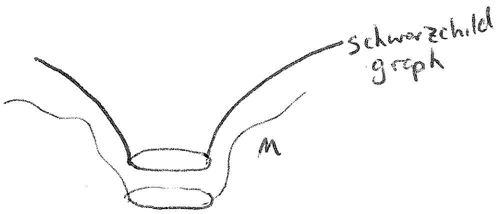
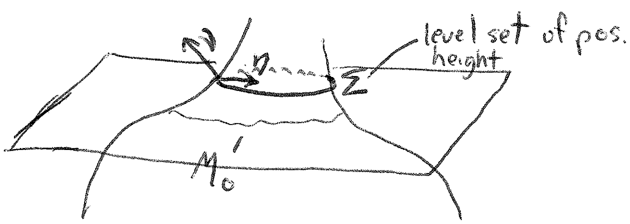


Fig 6



HYPERSURFACES WITH NON-NEGATIVE SCALAR CURVATURE

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Joint with Damin Wu.

If we have 2 d surface in $\mathbb{R}^3$, there are 2 types of curvature. Gauss curvature = product of principal curvatures ($K = \lambda_1\lambda_2$). Mean curvature = sum of principal curvatures. What are relations between them?

$K > 0$ implies that $\lambda_1, \lambda_2 > 0$. In the weak inequality case, $K \geq 0$, mean curvature could vanish.

ex: $f(x, y) = x^3$ (fig 1), λ_1 changes signs, $\lambda_2 = 0$. $K = 0$.

S.S. Chern - Lashof, 1950: If we assume $M^2 \subset \mathbb{R}^3$ is closed surface and if $K \geq 0$, then $\lambda_1, \lambda_2 \geq 0$ (up to orientation).

In higher dimension, have more types of curvature, so many generalizations. What if we assume that it has non-negative scalar curvature?

We're interested in manifolds with nonnegative scalar curvature. Weakest pointwise curvature. In GR, if we take a spatial slice in spacetime, induced M^n, g, k (k is 2nd fund form), then initial data set (satisfying constraint equations).

If we assume $k = 0$, i.e. time symmetric case, and assume dominant energy condition, then we have $R_g \geq 0$. So non-negative scalar curvature has meaning in GR.

Assume (M^n, g) can be isometrically embedded in $\mathbb{R}^{n+1}$. (strong condition).

Ex: (spatial) Schwarzschild: $g = \left(1 + \frac{c(n)m}{|x|}\right)^{4/(n-2)} \delta$ on $\mathbb{R}^n \setminus \{0\}$. (See fig 2) Middle sphere is minimal sphere. Ex: Any rotationally symmetric space can also be thus embedded.

Theorem 0.1 (H.,Wu)**.** *$(M^n, g) \subset \mathbb{R}^{n+1}$, C^{n+1} hypersurface. Assume M is either closed or asymptotically flat in the sense that $M = graph(f)$ outside a compact set and $|\nabla f| = o(1)$, (i.e. goes to 0, see fig 3). If $R_g \geq 0$, then $H \geq 0$.*

$A^i_j = g^{ik}\langle \nabla_{e_k} e_j, \nu\rangle = (\delta_{i_k} - \frac{u_i u_k}{1+|Du|^2})$ (more for this one?), where

$$\nu = (-\nabla u, 1)/\sqrt{1 + |\nabla u|^2}$$

is the unit normal (see fig 4). So $R(Du, D^2u) =$ 2nd symmetric polynomial of A.

$R(Du, D^2u) - R(Dv, D^2v)$ at a point $\nabla u = \nabla v$, then

$$(1)\quad = \frac{1}{\sqrt{1+|\nabla u|^2}}(H(\nabla u, \nabla^2 u)g^{ij}(\nabla u) - A^{ij}(\nabla u, \nabla^2 u) + H(\nabla v, \nabla^2 v)g^{ij}(\nabla v) - A^{ij}(\nabla v, \nabla^2 v))(u-v)_{ij}$$

This is linearization, so if want good things must put condition on mean curvature.

If $H \geq 0$ and $R_g \geq 0$, then the coefficient matrix of $(u-v)_{ij}$ is weakly positive definite (i.e. weakly elliptic), $= 0$ where $H = 0$.

For Schwarzschild hypersurface, the coefficient matrix is positive definite.

Theorem 0.2 (Time symmetric Penrose inequality)**.** *(M^n, g), $R_g \geq 0$, M^n, asymptotically flat, has an outermost minimal hypersurface Σ. Then the ADM mass $m \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{(n-2)/(n-1)}$, with equality iff M is Schwarzschild.*

So asymptotic behavior controlled by inner properties. $n = 3$, proved by Huisken- Ilmanen, and also Bray. $3 \leq n \leq 7$ proved by Bray-Lee. All dimensions, without rigidity and non-optimal constant by Schwartz and Jauregui.

For $n \geq 3$, Lam proved inequality if the manifold can be isom. embedded in $\mathbb{R}^{n+1}$. But does not prove equality case. Directly from Lam's proof, if equality holds, then $(M^n, g) \subset \mathbb{R}^{n+1}$ is scalar flat and Σ is connected round sphere. But there are plenty of these.

We now compare (M, g) with Schwarzschild graph with same size of round sphere. (assume round sphere has to be in the plane. Actually get in cylinder, at least, from Lam.) Lower graph till it hits M (doesn't hit at infinity), if it hits at interior point, then strong maximum principal (because $H \geq 0$!) since strong ellipticity given by comparison to Schwarzschild. Thus M is Schwarzschild.

Can be generalized. M_1, M_2, are asymptotically flat and scalar flat. If M_1 has $Hg^{ij} - A^{ij} > 0$ and can control at infinity (say $M_1 = M_2$ outside a compact set), then $M_1 \equiv M_2$ everywhere. This is only for ones that can be embedded in $\mathbb{R}^{n+1}$.

Theorem 0.3 (Corvino)**.** *If (M, g), asymptotically flat and $R_g = 0$, then there exists $(M, \tilde{g})$ such that $\tilde{g} = g$ in a compact set and $\tilde{g} =$ Schwarzschild outside a compact set and $\tilde{g}$ is scalar flat.*

Proof of Mean convexity theorem (if $R_g \geq 0$ then $H \geq 0$). Gauss equation $R_g = H^2 = |A|^2$, then $R_g \geq 0$ implies $H^2 \geq |A|^2$. If $H = 0$ then $|A| = 0$ (geodesic points). Thus Hessian of graphing function is 0.

Sacksteder: If M_0' is a connected component of geodesic points, then M_0' lies in a hyperplane tangent to M. (fig 6) Could be singular set. But if we move a bit away, get a nice smooth set.

Suppose H changes signs through M_0'. Then (for simplicity) Σ lies in the region $H \geq 0$.

Geometric inequality: H is the mean curvature of $\nu = (-\nabla, 1)/\sqrt{1 + |\nabla u|^2}$, H_σ is the mean curvature with respect to $\eta = (\nabla u/|\nabla u|, 0)$. We have $\langle \nu, \eta \rangle H H_\Sigma \geq R_g \geq 0$, and inner product is negative, thus $H_\Sigma \leq 0$. But that contradicts boundedness of Σ. Since it is close to M_0', it also cannot be bounded. $\square$