Cederbaum - Geometrostatics

Fig 1

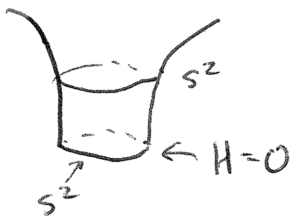


Fig 2

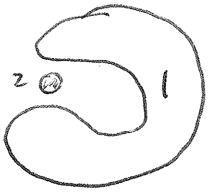
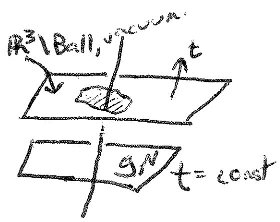
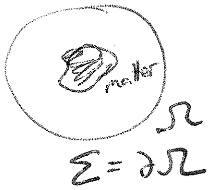


Fig 3



time indep. slices
isometric
$K=0$, i.e. time symmetric

Fig 4



GEOMETROSTATICS: THE GEOMETRY OF STATIC SPACETIMES IN GENERAL RELATIVITY

RAFAELA CARLA CEDERBAUM

Static spacetimes, i.e. ones that don't move.

Examples:

(1) Minkowski (special relativity) $\eta = -dt^2 + \delta$ where δ is Euclidean.
(2) Schwarzschild - outside of static, spherically symmetric ball.
$$ds^2 = -N^2c^2dt^2 + g$$
on $\mathbb{R}\times(\mathbb{R}^3\setminus 0)$, lapse N, where $g = (1+M/2r)^4\delta$ and $N = (1-M/2r)/(1+M/2r)$. (fig 1)
(3) Weyl class (axisymmetric)
(4) Andersson-Schmidt, '09, 2-body configuration. (fig 2) Proof by implicit function theorem. (not known if stable...) Infinitely many such based on matter types and such.

How could we talk about mass of each piece? center of mass of each piece? If we look at geodesics in such, how do they behave? Newtonian limits? Can we learn anything from Newtonian gravity? What can we learn as $r \to \infty$?

Take (L^4, ds^2) Lorentzian. Is "static" if exists a killing vector field X, timelike, hypersurface-orthogonality. (This is sometimes called static and stationary.) If it exists, we call it $X = \partial_t$. $N := \sqrt{-ds^2(X,X)}$ is the lapse. (fig 3) g is induced metric. We assume $ds^2 = -N^2c^2dt^2 + g$.

Such a system is isolated if $N \to 1$ as $r \to \infty$ and $g_{ij} \to \delta_{ij}$ as $r \to \infty$ and vacuum Einstein equations are satisfied outside some compact set.

All of this together is geometrostatic. All the examples earlier are.

Take vacuum Einstein, which is $\mathrm{Ric} = 0$, leads to static vacuum equations, which is $R = 0$ and $N \cdot \mathrm{Ric} = \nabla^2 N$ on a slice. These are equivalent to $\Delta N = 0$ and $N\mathrm{Ric} = \nabla^2 N$.

Geometrostatic are automatically Schwarzschildean, (Kennejich, O'Murchadha). and $g_{ij} \approx Schwarzschild_{ij} + 2M\vec{z}\cdot\vec{x}/r^3\delta_{ij} + O(1/r^3)$ where z is unique vector in $\mathbb{R}^3$ in wave harmonic coords (i.e. such that $\Box x^i = 0$). Also,
$$N = N(schwarzschild) - Mzx/r^3 + O(1/r^3).$$
$(M \neq 0)$. These are ADM mass and center of mass (M and z), $M = m_{ADM}G/c^2$, $z = z_{ADM} = z_{CMC}$.

(Note: Might not expect wave harmonic coords exist everywhere in exterior regions, but get $\Box x^i = 0$ iff $\Delta_\gamma x^i = 0$. Also $\gamma = \delta + O(1/r^2)$. And then things behave well under Euclidean motions.)

But these are only mass of whole configuration, not for piece. These more local masses we call quasi-local mass and center of mass. Do change of variables. $U = c^2 \ln N$ and $\gamma = N^2 g$. Can translate asymptotic results to these. $U = -mG/r - mGz \cdot x/r^3 + O(1/r^3)$. (Pseudo-Newtonian potential) Can get $\mathrm{Ric}_\gamma = 2/c^4 dU \otimes dU$ and $\Delta_\gamma U = 0$. Somewhat similar to Newtonian.

In Newtonian case,

$$\begin{aligned} m_N = \int_{\mathbb{R}^3} \rho dV &= \frac{1}{4\pi G} \int_\Omega \Delta V dV \\ &= \frac{1}{4\pi G} \int_\Sigma v(U) d\sigma \end{aligned}$$

(fig 4)

$$\vec{z}_N = \frac{1}{m} \int \rho \vec{x} dv = \frac{1}{4\pi Gm} = \int_\Sigma (v(U)x - Uv(x)) d\sigma$$

Define (PN is pseudoNewtonian)

$$m_{PN} := \frac{1}{4\pi G} \int_\Sigma v(U) d\sigma,$$

with respect to γ and

$$\vec{z}_{PN} = \frac{1}{4\pi Gm} \int_\Sigma (v(U)x - Uv(x)) d\sigma.$$

Thanks to Laplace equation for U, this is well defined for any surface Σ containing the matter. For any large enough, $m_{PN}(\Sigma_\infty) = m_{ADM}$ and similar for center of mass. It's nonnegative under strong energy condition.

But now, we can take Σ only containing one piece of the matter. If we have two bodies, we get the PN masses add to the total mass! $M_1 z_1 + M_2 z_2 = M_{tot} z_{tot}$. No potential energy or kinetic energy in some sense. If we take the Newtonian limit in some sense, these converge to the Newtonian masses and center of masses.