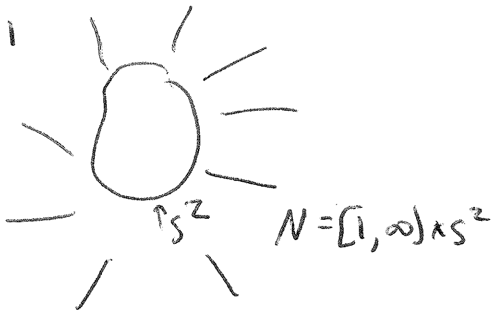
Lin - Ricci Flow and prescribed scalar curvature.

Fig 1



ON HAMILTON'S RICCI FLOW AND BARTNIK'S CONSTRUCTION OF METRICS OF PRESCRIBED SCALAR CURVATURE

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(S^2, g_1), R is scalar curvature. $r = \int R d\mu / \int d\mu = 2$, i.e. mean scalar curvature. Ricci potential f is solution to $\Delta f = R - r$, which we know is solvable. Define $M_{ij} = D_i D_j f - \frac{1}{2}(R - r) g_{ij}$ which is trace free part of the Hessian of f.

The modified Ricci flow is $\partial_t g = (r - R) g_{ij} + 2 D_i D_j f = 2M_{ij}$. On surfaces with any arbitrary metric $g(1, \cdot) = g_1$ arbitrary, the solution exists for all time, and $g(t) \to g_{S^2}$ exponentially and $M_{ij} \to 0$ exponentially. Under this modified flow, volume is preserved.

Scalar curvature problem: $N = [1, \infty) \times S^2$, $\bar{R} \in C^\alpha(N)$, Question: Can we find a asymptotically flat (AF) metric $\bar{g}$ with scalar curvature $\bar{R}$?

$dt^2 + t^2 g_{S^2}$ is flat. Let $\bar{g} = u^2(t, x) dt^2 + t^2 g(t)$. We could hope this is AF. 2nd fundamental form $h_{ij} = \frac{1}{u}(\frac{1}{t}\bar{g}_{ij} + M_{ij})$ and so mean curvature is $H = \frac{2}{tu}$. Also, u satisfies

$$t \partial_t u = \frac{1}{2} u^2 \Delta_g u + Au + Bu^3,$$

where A, B are functions. (If we treat $\bar{R}$ as given, the problem becomes this parabolic equation.)

Let $w = u^{-2}$, then it becomes instead

$$t \partial_t w = \frac{1}{2w} \Delta_g w + \frac{3}{2} u \nabla u \nabla w - (t^2/2|M|_g^2 + 1) w + R/2 - t^2/2\bar{R}.$$

(fig 1 - We flow out, becoming asymptotic to S^2.) We can specify the boundary metric (g_1) and the mean curvature

$$H = \frac{2}{tu}\bigg|_{t=1} = 2/u(1, \cdot).$$

This was originally done by Bartnik in 1993.

Theorem 0.1. *Assume $\bar{R} \in C^\alpha(N)$ and*

$$K := \sup_{1 \le t < \infty} \left[-\int_1^t (R/2 - \tau^2/2\bar{R})_* e^{\int_1^\tau s|M|^{*2}/2 ds} d\tau \right] < \infty.$$

Then for any function $\phi \in C^{2,\alpha}(S^2)$ satisfying $0 < \phi < 1/\sqrt{K}$, there exists unique positive solution $u \in C^{2+\alpha}(N)$ of the equation above with initial condition $u(1, \cdot) = \phi(\cdot)$. (Sub and super stars are infimums and supremums)

If in addition, for all $t \geq 1$ we have the fall off estimates $\|\bar{R}t^2\|_{\alpha,I_t} \leq \frac{c}{t}$ and $\int_1^\infty |\bar{R}|^* t^2 dt < \infty$, then $\bar{g}$ is AF and ADM mass is finite and

$$m_{ADM} = \lim_{t\to\infty} \frac{1}{4\pi}\int \frac{t}{2}(1-u^{-2})d\sigma.$$

Why is ADM mass that? $g(t) \to g_{S^2}$ exponentially. On each leaf, $\{t\}\times S^2$ is nearly round. Thus

$$\begin{aligned} m_{ADM}(N) &= \lim_{t\to\infty} m_H(\{t\}\times S^2) \\ &= \lim_{t\to\infty} \sqrt{A(S_t)/16\pi} - \frac{1}{16\pi}\int H(t)^2 d\sigma_t \\ &= \lim 4\pi\sqrt{t^2/16\pi} - \frac{1}{16\pi}\int \frac{4}{t^2}u^2t^2 d\sigma \\ &= \frac{t}{2}(1 - \frac{1}{4\pi}\int u^{-2}d\sigma) \end{aligned}$$

which is what we wanted.

Hawking mass in non decreasing with respect to t.

Hawking mass converges to ADM mass by a proof by Shi, Yuguang, Wang, Guofang Wu and Jie, since nearly round.

Someone asked if you need to make $g(t)$ satisfies Ricci flow. We just needed $u^2dt^2 + t^2e^{2f}g_{S^2}$ for something similar. How arbitrary could they be? There are some possibilities, but the conditions are complicated. It's simple for Ricci flow. Bartnik's original paper, he used $t^2g_{S^2}$. In general, these are large t conditions on f, but there can be conditions for t small for things like black hole boundaries.

Does inverse mean curvature flow "go through the levels?" (i.e. flow out to $[0,\infty)\times M$?) Unknown.

It gets complicated immediately if try to get it to solve the Einstein constraints in general, not just prescribed scalar curvature.

Might be able to generalize to higher dimensions.

With more conditions, can allow initial mean curvature to be 0.

What would happen if we evolve this via vacuum Einstein? No results for how any of these relate.

If we start with (S^2, g_{S^2}), we know Ricci flow does nothing. If $\bar{R}\equiv 0$, then we know $\bar{g}$ is Schwarzschild. If the starting is not S^2, we don't always get Schwarzschild. But it does make $K=0$, and so we can have any initial condition and still solve the equation.