Integrability meets Control Theory: Harmonic Maps in General Relativity

We provide a framework for analyzing axially symmetric harmonic maps on $\mathbb{R}^3$ with symmetric target spaces $G/K$. Drawing on results from analysis to Lie theory to geometry, we give a complete and rigorous proof that, all such maps are completely integrable. We further demonstrate that new solutions to the harmonic map equations can be generated from a given seed solution, using a dressing or vesture method. This unifies the integrability of theories including chiral field models, nonlinear $\sigma$-models, Yang-Mills and Einstein electrovacuum equations in the general context of harmonic maps.

Utility of the vesture method is made concrete by generating $N$-solitonic harmonic maps into a noncompact Grassmann manifold $G = SU(p, q)$. We demonstrate a special case by deriving Kerr and Kerr-Newman solutions from the Minkowski initial seed for the Einstein vacuum and Einstein-Maxwell cases, respectively. In performing an asymptotic analysis, these solutions are shown to be in the hyperextreme sector of the corresponding parameters, suggesting constraints on the dressing mechanism. We indicate the possibility of using this analysis to control the resulting $N$-black hole configurations in this setting.