Shabnam Beheshti - Integrability

Fig 1

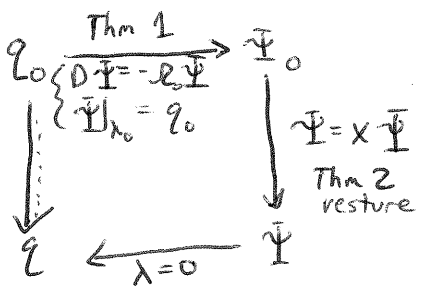
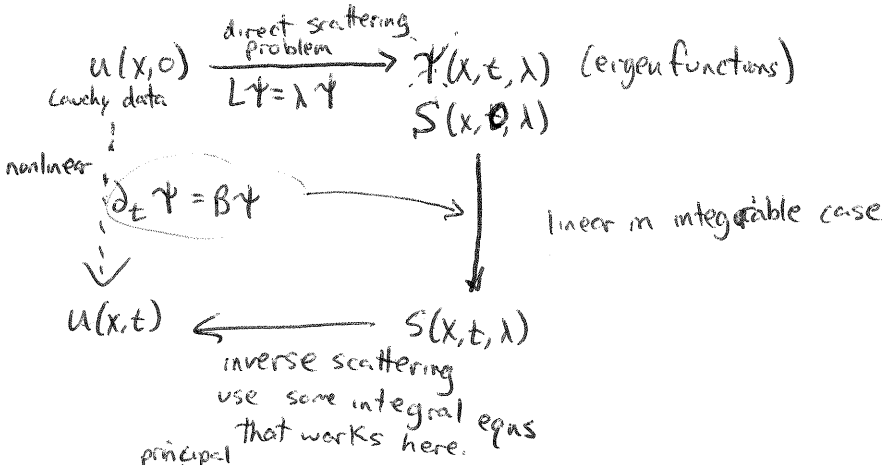
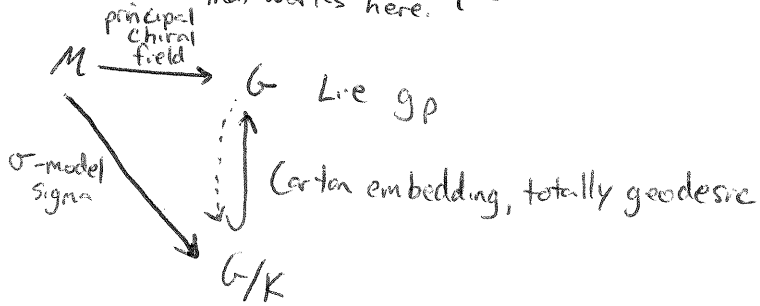


Fig 2



$\lim_{x\to\infty} \Psi = \lim_{x\to-\infty} \Psi$

Fig 3



INTEGRABILITY MEETS CONTROL THEORY: HARMONIC MAPS IN GR

SHABNAM BEHESHTI

Joint work with S. Tahvildar-Zadeh.

A question from her postdoc years: Is there a way to control the solutions of inverse scattering mechanism (ISM) in GR?

Her goes is to explore interconnections between gravitation (Einstein vacuum equations (EVE) and electromagnetism (EM)), Integrable PDE and Harmonic maps.

Our goal in this talk is to understand Figure 1. Theorem 1 is an integrability problem, Theorem 2 is "dressing" the solution. Our plan is:

(1) Classical Integrability/ISM
(2) ISM for EVE
(3) Harmonic Maps
(4) Results!
(5) Main elements of P
(6) Example
(7) Control Elements - Future Work

1. A simple beginning: Linear evolution problems, like $u_t = cu_x = 0$. For this problem, $u(t,x) = f(x-ct)$, a solution, is a traveling wave form. Or for $u_{tt} + \Delta u = 0$, we have solutions like $u(t,x) \approx e^{i(k\cdot\vec{x}+|k|t)}$, which is a planar wave. Certain non-linear PDEs still have this wave form of solutions. We have the superposition principle of traveling/plane wave solutions (in linear evolution problems). On the other hand, nonlinear equations don't generally posses this feature.

Kurskal - Zubusky '65: KdV problem has "hidden" wave solutions. This prompted study of integrable PDE. For nonlinear equations, wave solutions are solitons, i.e. idealized solutions that act like waves.

Definition 0.1. Idea of integrable: A PDE or system of PDE is said to be integrable if 1. You can "integrate it" (explicit solutions) a la Liouville. 2. an infinite family of conservation laws. 3. a Lax pair exists. 4. ISM applies. (one of these) We will use 3.

We will use the equation $u_t = F(u, u_x, u_{xx}, \cdots)$. Associate to it a corresponding overdetermined linear scattering system for an isospectral family $L(t,u)$ whose eigenfunctions ψ satisfy a secondary equation, $\partial_t\psi = B\psi$ (fig 2). Isospectral then implies compatibility condition $L_t = [B, L]$. We then say L, B is a Lax pair. (defined in '68 by Lax)

For Example: KdV equation is $u_t + 6uu_x + u_{xxx}$, $L = -\partial_x^2 + u$, B some function of u, u_x, u_{xx}. Not clear how to find, but can find sometimes.

If I can find this pair, there is a process I can use to evolve the equations. This transforms nonlinear evolution to *linear* existence!

Why would we do this perverse process? There are many equations for which this all applies and works out.

How does this apply for EVE?

2. ISM/Integrability for EVE.

Belinski-Zakharov in '78/'79 - got integrability for static, axisymmetric EVE. They assumed existence of 2 commuting Killing fields. The metric can then be rewritten as

$$g_{\mu\nu}dx^\mu dx^\nu = f(\rho, z)(d\rho^2 \pm dz^2) + \tilde{g}_{ab}dx^a dx^b$$

Also, EVE can be reexpressed as some system in $\tilde{g}$ + some equations for f which can be solved explicitly by "quadrature" if $\tilde{g}$ is given. Then reexpress this system in $\tilde{g}$ as our PDE of interest. This is actually an integrable system. To do this you have to generalize idea of derivatives.

They write the overdetermined linear system as $D_1\psi = U\psi$, $D_2\psi = V\psi$ where U, V are matrices, D_1, D_2 are matrix operators. Where $D_j = \partial_j - p(\rho, z, \lambda)\partial\lambda$. (derivative in spectral component) We can then start this mechanism to find solutions.

3. Harmonic: On the other hands, the 2 Killing field reduction of the EVE gives the Ernst equation for a complex potential $\epsilon(\rho, z)$

$$(\epsilon + \bar{\epsilon})\Delta\epsilon + 2\nabla\epsilon \cdot \nabla\epsilon = 0$$

but this is a harmonic map equation!

Definition 0.2. $f : (M, g) \to (N, h)$ is harmonic, if it is a critical point of the Dirichlet energy, $\int_D \frac{1}{2}\mathrm{tr}_g f^* h$. If M is 1 dimensional, this is really just geodesics. If 2 d, generalizes harmonic.

Ex: If we look at $f : \mathbb{R}^3 \to SL(2, \mathbb{R})/SO(2) \simeq \mathcal{H}_{\mathbb{R}}$, and the metric on $\mathbb{R}^3$ is $ds^2 = dz^2 + d\rho^2 + \rho d\phi^2$ then get harmonic map function. Metric on the upper half is $ds^2 = (\mathrm{Re}\ \epsilon)^{-2}|d\epsilon|^2$. We then recover the Ernst Equation.

4. Results

Main Point: EVE + 2 Killing fields are integrable because (certain) harmonic maps are integrable.

Theorem 0.3 (Theorem 1)**.** *Let G be a real semisimple Lie group and K a maximal compact subgroup. Then any axially symmetric harmonic map from $\mathbb{R}^3 \to G/K$ is integrable. (i.e. harmonic map is compatibility condition for a findable Lax pair).*

Theorem 0.4 (Theorem 2)**.** *If G/K can be realized in terms of its involutions as conjugations by a common element, then solutions can be dressed. (I.e. in Fig 1, you can find your X which makes it work... she said more but hard to write.)*

5. Elements of proof(s).
   (1) Identify your field equations of interest with those of a harmonic map. (see fig 3)
   (2) Rewrite harmonic map equations as a nonlinear Hodge system. Metric on G/K comes from maurer? - Cartan form on N, $w = -dgg^{-1}$ pull back to on on M, $W = -dqq^{-1}$. the pullback of the structure equations give something like $dW + W \wedge W = 0$, $d(\rho \star W) = 0$ (comes from harmonic map equations, and this is the Hodge star.)
   (3) Exhibit a Lax pair whose compatibility condition is the Hodge system. The trick is rewriting the above equation as $dq = -Wq$ and then generalize this to $D\Psi = \Lambda\Psi$ by embedding spectral parameter in D, etc. There are some nice geometric methods for all this. This completes theorem 1.
   (4) Construct dressing matrix X possessing appropriate symmetries. (Want to ensure q in Fig 1 is still harmonic map into same symmetric space.) Involves $G_{reality}$ and involutive symmetries.
   (5) $X = I + \sum_{k=1}^{2N} \frac{R_k}{\lambda - \lambda_k}$ Reduces problem to finding solutions to a linear algebraic system on R_k. That completes the vesture (dressing) problem.

Example: Dressing with a simple pole can leave you naked!

$$\begin{aligned} H &= SL(p+q, \mathbb{C}) \\ G &= SU(p,q) \\ K &= S(U(p) \times U(q)) \end{aligned}$$

Map from $H \to G$ is $\tau(g) = \Gamma g^{-\star}\Gamma$, map from $G \to K$ is $\sigma(g) = \Gamma g \Gamma$. Complex Grassmannian , $G/K = \{q \in G | q\Gamma q \Gamma = I\}$. We have $p = 1, q = 1 \to \mathbb{H}_{\mathbb{R}}$ is EVE, and $p = 1, q = 2 \to \mathbb{H}_{\mathbb{C}}$ is EM. Here Γ is block diagonal matrix with $I_{p\times p}$ and $-I_{q\times q}$.

Choose $q_0(\vec{x}) = I_{4\times 4} : \mathbb{R}^2_+ \to SU(2,2)/S(U(2) \times U(2))$ (Minkowski seed). This implies $\Psi_0(\vec{x}, \lambda) = I_{4\times 4}$. Dress q_0 with one pole by fixing $\omega = is$, $s > 0$. This implies

$$X(\vec{x}, \lambda) = I + \frac{R_1}{\lambda - \bar{\lambda}_1} + \frac{R_2}{\lambda - \bar{\lambda}_2}$$

where λ_i are roots of $\lambda^2 - 2(z - \omega)\lambda - s^2$. Linear algebra rewrite:

$$\begin{bmatrix} \frac{|\alpha|^2 - |\delta|^2}{\lambda_1 - \bar{\lambda}_1} & \frac{|\alpha|^2 + |\delta|^2}{\lambda_1 - \bar{\lambda}_2} \\ \frac{|\alpha|^2 + |\delta|^2}{\lambda_2 - \bar{\lambda}_1} & \frac{|\alpha|^2 - |\delta|^2}{\lambda_2 - \bar{\lambda}_2} \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} = \begin{bmatrix} -v_1^* \\ -v_2^* \end{bmatrix}$$

New solution $q(\vec{x}) = X(\vec{x}, \lambda = 0$. $q_0(\vec{x}) = I_{4\times 4} = \frac{1}{\lambda_1} u_1 v_1^* - \frac{1}{\lambda_2} u_2 v_2^*$. Change Weyl coord $\vec{x} = (\rho, z)$ to Boyer-Lindquist r, θ.

$$q = \begin{bmatrix} 1 + \frac{8|\alpha|^2|\delta|^2 a^2 s^2}{F} & \frac{1}{F}\{-4s\alpha\bar{\delta}(iA(r-m) + Bs\cos(\theta))\} \\ \frac{-1}{F}\overline{\{\cdot\}} & 1 + \frac{8|\alpha|^2|\delta|^2 a^2 s^2}{F} \end{bmatrix}$$

For $A = |\alpha|^2 - |\delta|^2 = s$, $B = |\alpha|^2 + \delta|^2 = a := \sqrt{m^2 + s^2}$, $\alpha\bar{\delta} = m/2$. q recovers the (naked) Kerr solution!