Moore

Fig 1

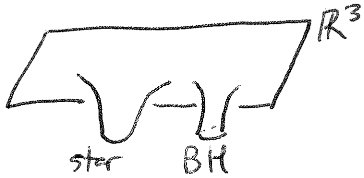


Fig 2

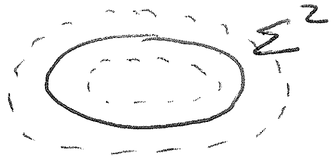


Fig 3

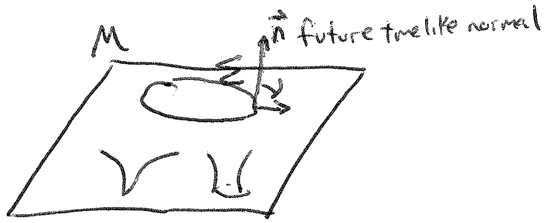


Fig 4

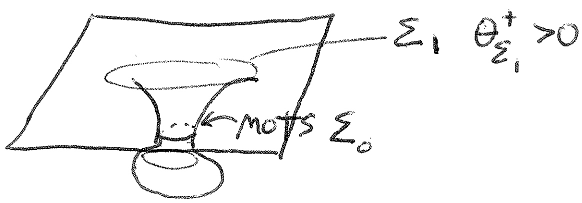
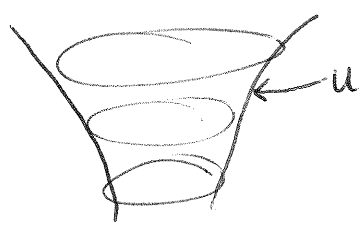


Fig 4



MARGINALLY OUTER TRAPPED SURFACES AND "NULL" MEAN CURVATURE FLOWS

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Joint with T. Bourni.

Her goal is to use geometric evolution equations to solve problems in GR.

GR : (L^4, h), with h Lorentzian $(-+++)$. We will deal with isolated gravitating systems, and thus asymptotically flat (AF) manifolds. A standard approach is to decompose as $(M^3, g, K) \subset (L^4, h)$, with the induced metric g Riemannian, and K the 2nd fundamental form, which measures how much M curves in L. Think of the slice M as one instant of time. (figure 1)

Black hole horizon: The event horizon is where light can't escape. But these are global spacetime objects. So, to find them, we instead look for "trapped surfaces."

A trapped surface is where the light emitted from a Σ^2 is taking up less area (figure 2), i.e. light rays are convergent. The outermost part is a marginally trapped surface or apparent horizon. This is the boundary of the trapped region, but it can be computed (found) locally.

To be more rigorous, (see fig 3) get $l^{\pm} := \vec{n} \pm \nu$ and $\vec{H}_\Sigma : -H\nu - Pn$, where $H = \mathrm{div}_M \nu$ and $P = \mathrm{tr}_2 K = \mathrm{tr}_M k - K(\nu, \nu)$. Then let $\theta^{\pm} = \langle \vec{H}, l^{\pm} \rangle$. These are outer/inner null mean curvature or null expansions. (They represent how much the light rays from Σ expand or contract in area in null directions.

If $\theta^+_\Sigma = 0$, then Σ is marginally outer trapped surface (MOTS), and similarly for MITS. [This is because the outward light rays are not expanding in area.] If $K = 0$, MOTS are also minimal surfaces. This is a Lorentzian version of minimal surfaces.

This has been investigated by Andersson-Metzger, etc. Existence has been proved of MOTS under physically appropriate trapping assumptions. Elliptic methods don't give the properties want, however, so we try parabolic methods, because they can tell what the MOTS actually look like. So take a piece like Σ_1 in Fig 4, and flare in to find the MOTS. Or do one inside and flare out to evolve to it. The right speed for this flow is the inverse null mean curvature flow.

The definition for this flow is $F_0 : \sigma \subset M^3$, where $\partial_t F = \frac{1}{\theta^+}\nu(x,t)$ and $F(\cdot, 0) = F_0(\cdot)$. We also assume $H + P|_{\Sigma_0} > 0$. This is a generalization of $K \equiv 0$ inverse mean curvature flow (IMCF).

A round sphere under this flow grows exponentially fast and expands forever.

The linearization of this flow is $L \approx \frac{1}{\theta^{+2}}\Delta + \cdots$. Since $\theta^+ > 0$, this is strictly parabolic, and so we get existence.

$$\partial_t \theta^+ = \frac{1}{\theta^{+2}}\Delta\theta^+ - |D\theta^+|^2/\theta^{+3} - \cdots$$

etc. This is a reaction diffusion equation.

We then get : $\theta^+ \leq \max(\theta^+_{\partial\Sigma_t \cap B_R} C(n, R, \|K\|))$ where C is as $1/R$ as $R \to \infty$. We expect this null mean curvature will go to 0 in finite time.

On torus, we get singularity at center point when we evolve. But if starting set is star shaped, then there is long time existence, and it will approach a sphere. So, if we want long time existence, we need a change of topology...

We can use the level set method, which has been used for IMCF: Let $u : M \setminus E_0 \to \mathbb{R}$, $u \geq 0$ and $\Sigma_t \partial\{u < t\}$. For the round sphere in $\mathbb{R}^n$, the level set function is $u(x) = n \ln(|x - x_0|)/R_0)$. (fig 4)

The level set function must satisfy

$$\mathrm{div}_M(\nabla u/|\nabla u|) + (g^{ij} - \nabla^i \nabla^j u/|\nabla u|^2)k_{ij} = |\nabla u|.$$

The first term is H, the second is P, and the right hand side is inverse speed.

We use the approach of elliptic regularization, which is equivalent to solving Jang's equation. We use regularity results of geometric measure theory. From that, get existence of locally Lipschitz solution to this level set equation, i.e. $u \in C^{0,1}_{loc}(M \setminus E_0)$.

We then have the variational characterization of weak solutions: The evolving surface $\Sigma_t := \partial E_t$ must always be "outward optimizing." (Like outward minimizing), i.e. it minimizes "area and bulk energy term P",

$$J(E_t) : +|\partial E_t| + \int_{E_t \setminus E_0} P$$

on the outside i.e. for any $F \supset E_t$ such that $F \setminus E_t \subset\subset M$. There will be a jump discontinuity in volume at the singularity, essentially, to the next minimizer, then the flow will continue. The torus jumps to a topological sphere, for instance, once it is more efficient to be a sphere than a torus.

Thm: Weak existence: Let (M, g, K) be AF, with $\mathrm{tr}_M K \geq 0$. Then we have a weak solution of our inverse null mean curvature flow with any initial condition $E_0 \subset M$, i.e. we can start our flow from any trapped surface.

Thm: If $\theta^+|_{\partial E_0} < 0$, (the solution will jump instantly, since this is not outward optimizing initially), then $\theta^+|_{\partial\{u>t\}} \equiv 0$ i.e. $\partial\{u > t\}$ is a smooth MOTS.

This was for flowing *out* to MOTS. Now for flowing in.

We define the null mean curvature flow, $F_0 : \Sigma^2 \to M^2$ where $\partial_t F(x,t) = -\theta^+\nu = -(H + P)\nu$, and $H + P|_{\Sigma_0} > 0$.We proceed similarly, and solve th level set formulation. This proof is even more like Schoen and Yau's use of Jang's equation. We get the variational characterization, perform elliptic regularization, add a capillarity regularity term, and then send that to 0. The solution will already blow up to the MOTS. We only need the elliptic approach here, even though it is a parabolic flow.

If there is no MOTS to be found, the solution will disappear.