

# Schoen - localized Solutions

Fig 1

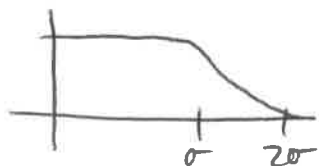
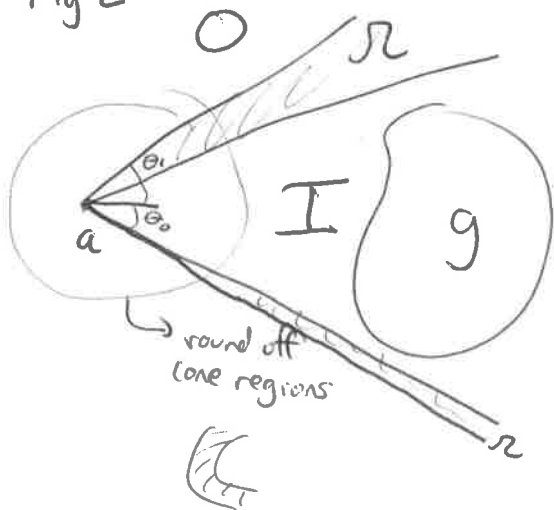


Fig 2



# CONSTRUCTION LOCALIZED SOLUTIONS OF THE EINSTEIN CONSTRAINT EQUATIONS

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Take  $(M^n, g, k)$ . Then the vacuum constraints (\*) are  $R_g - |k|_g^2 + (\text{tr}_g k)^2 = 0$  and  $\text{div}(k - (\text{tr}_g k)g) = 0$ . This is  $n + 1$  equations if we are in  $n$  dimensions. If we are in a spacetime  $\mathcal{S}$ , and  $\text{Ric} = 0$ , then the constraint equations are implied for any spacelike slice  $M$ .

The Cauchy problem is the converse of this. If we have the initial data set, then there is, at least locally, a spacetime with that as a slice.

Again, there are  $n + 1$  equations for the constraints, but  $n(n + 1)$  unknowns, and so we should expect to have lots of solutions.

He wants solutions that are asymptotically flat (AF). Definition: Assume there is a coordinate system near infinity so that you can write  $M = K \cup \text{Ext}$  where the exterior region  $\text{Ext} \simeq \mathbb{R}^n \setminus B^n$ . Our coordinates will be  $x^1, \dots, x^n$ . Then, a slice is AF if the coordinates are such that  $g_{ij} = \delta_{ij} + o_2(|x|^{-p})$  (the subscript 2 means the second derivatives also fall off at the same rate) and assume  $k_{ij} = o_1(|x|^{-p-1})$ .

If we assume strong enough asymptotics, for instance if  $p > (n - 2)/2$ , we can assign ADM energy and linear momentum to AF manifolds.

$$E = \frac{1}{2} \lim_{\sigma \rightarrow \infty} \int_{S(\sigma)} (g_{ij,j} - g_{jj,i}) \nu^i d\sigma$$

$$P_i = \lim_{\sigma \rightarrow \infty} \int_{S(\sigma)} (k_{ij} - \text{tr}(g)g_{ij}) \nu^j d\sigma$$

The condition  $p > (n - 2)/2$  guarantees that  $R$  is integrable, which plays into this being finite and well defined.

The positive energy theorem (PET) says that  $E \geq 0$  and  $E = 0$  only if  $(M, g, k) \subset \mathbb{R}^{n+1}$ , i.e. it is a slice of Minkowski [technically this is proved only for  $n \leq 7$ ]. Thus we can't find solutions which have, for instance, compact support, since then the energy would be zero.

But....

What are natural asymptotics? The most natural are the ones given by the exact solutions. So consider  $(g, k) \approx$  Schwarzschild near infinity. The  $t = 0$  slice of Schwarzschild, has that  $k = 0$  and  $g = (1 + E/2|x|^{n-2})^{4/(n-2)}\delta + o_2(|x|^{1-n})$ . Those then seem like reasonable asymptotics. We could also take non-constant  $t$  slices that have nontrivial energy and linear momentum.

The conformal method does come up with solutions with asymptotics like this.

For  $k \equiv 0$ , we say the slice is time symmetric, and then the constraints are just  $R(g) \equiv 0$ .

**Theorem 0.1** (Corvino '00). *Start with  $g$  with the above asymptotics. For  $\sigma \gg 1$ , there exists a new solution  $\hat{g}$  which is exactly  $g$  in ball of radius  $\sigma$  and is exactly Schwarzschild (with some energy and center) outside of  $B_{2\sigma}$ . (We call this Schwarzschild metric  $g_{\hat{E},x_0}$ .) Also,  $\hat{g}$  is as smooth as  $g$ , and the energy of the Schwarzschild  $E$  is very close to the energy of  $g$  (it actually converges as  $\sigma \rightarrow \infty$ ).*

There is a spacetime version of this result, but it is more complicated (we need that it is exactly Kerr near infinity, for instance).

Outline of proof: Construct a  $\tilde{g}$  which doesn't satisfy the constraints, but has the center and asymptotics, i.e. something like  $\xi_\sigma g + (1 - \xi_\sigma)g_{\hat{E},x_0}$  for a smooth cutoff function  $\xi_\sigma$  (see fig 1).

We then find a perturbation  $h$  which is small and supported in  $B_{2\sigma} \setminus B_\sigma$  with  $h \approx d^n$  where  $d$  is the distance to the boundary of the annulus, i.e. it vanishes at the boundary to high order, which is important. This perturbation is such that  $\hat{g} = g + h$  and  $R(\hat{g}) = 0$ .

We then get  $R(\tilde{g} + h) = R(\tilde{g}) + Lh + Q(h, \partial h, \partial^2 h)$ . We invert the linear part,  $Lh$ , and then use Picard iteration and fixed point theory to get a solution. Here,

$$Lh = h^{ij}{}_{;ij} - h^{ij}\tilde{R}_{ij} - \Delta_{\tilde{g}}\text{tr}_{\tilde{g}}h.$$

Note that  $L : \text{Sym}_2 \rightarrow C^\infty$ . We need to invert it to make sure we have the fast decay to boundary. In order for  $L$  to be surjective, need the formal adjoint to be injective. So, we have

$$L^*f = f_{;ij} - f\tilde{R}_{ij} - (\Delta f)\tilde{g}_{ij}.$$

We're very close to Euclidean space since we're AF. In that case, we would have  $L_\delta^* = f_{;ij} - (\Delta f)\delta_{ij}$ . Notice the  $\ker L_\delta^*$  are exactly the functions with vanishing hessian, i.e.  $\text{span}\{1, x^1, \dots, x^n\}$ . In general, the kernel is the obstruction to being able to do this gluing. This kernel is a  $n+1$  parameter family, but there are  $n+1$  parameters of Schwarzschild to play with (mass and center of mass), and so they play with each other to work out.

This kind of gives a localization of solutions of the constraint equations.

Suppose we take a solution with good asymptotics,  $g$ , AF and with  $R(g) \equiv 0$ . Let  $U = \{p \in M : \text{Ric}_p \neq 0\}$ . In 3 dimensions, this is where the full tensor is 0, but in higher dimensions it may not be.

**Proposition 0.2.** *If  $g_{ij} = \delta_{ij} + o_2(|x|^{2-n})$  then either  $g$  is flat or*

$$\liminf_{\sigma \rightarrow \infty} \sigma^{1-n} |U \cap S(\sigma)| > 0.$$

*Thus, the set  $U$  can't be a strip or something that has vanishing angle. There has to be some kind of fixed angle at infinity that is nonzero.*

The energy can also be written

$$E = -c(n) \lim_{\sigma \rightarrow \infty} \sigma \int_{S(\sigma)} Ric(\nu, \nu) d\Sigma.$$

Proof: for  $\sigma \gg 1$ , that integral for  $E$  has to be strictly non-zero. Also,  $Ric$  decays as  $\sigma^{-n}$ . We have

$$c\sigma^{1-n}|U \cap S(\sigma)| \geq \left| c(n) \int_{S(\sigma)} Ric(\nu, \nu) d\Sigma \right| \geq \epsilon_0 > 0$$

So we can only have Ricci flat on a relatively small set.

**Theorem 0.3.** *With A. Carlotto. Start with  $(M, g)$  with  $R(g) = 0$  and  $g = \delta + o_2(|x|^{2-n})$  (i.e. it decays like Schwarzschild). If  $0 < \theta_0 < \theta_1 < \pi$  and  $q < n-2$ , then there exists some large  $\Lambda$  such that if  $a \in \mathbb{R}^n$  with  $|a| > \Lambda$  and there exists a  $\hat{g}$  with  $R(\hat{g}) = 0$  with  $\hat{g} = g$  in  $I$  (see figure 2) and  $\delta$  (i.e. Minkowskian) in  $O$ , the exterior portion. In the transition region  $\Omega$ ,  $\hat{g} = \delta + o_2(|x|^{-q})$  (i.e. you have to give up some fall off rate).*

So we can localize solutions in a *cone*, even for cones of arbitrarily small angle, though it is impossible to localize solutions generally.

The proof works fairly similarly to the Corvino case, but transition region is noncompact so have to worry about the decay at infinity. We came upon this while trying to show that you *can't* find solution that is Euclidean on a half-space. Oops! (This may actually still be true if we assume  $q = n-2$ , but that's another, harder story.)

- (1) There exists a solution of (\*) supported in an arbitrarily small cone.
- (2) Most of the [ADM] energy is in the transition region  $\Omega$ , which suggests that the fall off must be worse, as it is in the theorem. However, the overall mass is almost the same.
- (3) This method then gives new initial data for the  $n$ -body problem different from the gluing of Chrusciel, Corvino and Isenberg. Here we can glue by just doing two of these and putting cones near (but not overlapping) each other.

There is also a spacetime version of this theorem.

The main new idea is about how to get the decay correct. Take  $H_{k,-q}(\Omega)$ , functions with bounded support in  $\Omega$  with norm

$$\|u\|_{k,-q}^2 = \sum_{i=1}^k \sum_{|\alpha|=i} \int_{\Omega} r(x)^{-n+2q+2i} |\partial^\alpha u|^2 d\mu,$$

where we pick some  $r(x) > 0$  with  $r(x) = |x|$  outside a ball. Then we have the duality  $H_{0,-q-2}^*(\Omega) = H_{0,q+2-n}(\Omega)$ . If  $q < 2-n$ , this is a negative exponent. This will be the domain of  $L^*$ , defined as before, and so this guarantees we killed the obstructions, since the kernel of  $L^*$  on this space is trivial (essentially because 1

and  $x^i$  don't go to zero). So in some sense this is easier than Corvino, since we don't have to worry about these. There is no obstruction!

The "Basic Estimate": if  $u \in H_{0,q+2-n}$  (call  $p = q + 2 - n$ ). Then

$$\|u\|_{2,-p} \leq C \|L^*u\|_{0,-p-2}$$

with no boundary condition on  $u$ , and without [the usual] lower order term!

Claim: though you can't localize solutions, you can almost do it.

Is the center of mass near to that of the original? Seems like it should be, but this hasn't been done.

Do think this is false for  $q = n - 2$ ? Well, we can't control the *constant* multiplying  $|x|^{n-2}$ , but we don't know.