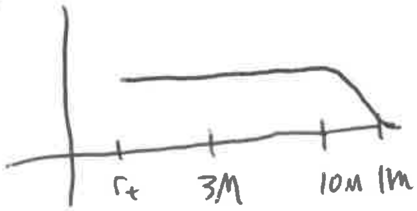
Blue - Decay of fields

Fig 1:



FURTHER TOPICS IN DECAY OF FIELDS OUTSIDE BLACK HOLES

PIETER BLUE

This is a continuation from the last talk.

1. Energy generation and strengthening (Continued)

Last time, we introduced analogs of the stress energy tensors for various model problems. We have the energy

$$E_X(\Sigma) = \int_\Sigma T_{ab} X^b d\nu^a$$

For the null geodesics, we define it instead as $\dot\gamma_a X^a|_{\gamma\cap\Sigma}$. We define

$$Bulk_X(\Omega) = \int_\Omega T_{ab} \nabla^{(a} X^{b)} d\mu$$

or

$$\int_{\gamma\cap\Omega} \dot\gamma_a \dot\gamma_b \nabla^{(a} X^{b)} d^4\mu$$

for null geodesics γ. Here, $\nabla^{(a} X^{b)}$ is the deformation tensor, which is also equal to $-\frac{1}{2} L_X g$.

A symmetry of a PDE is a differential operator taking solutions to solutions.

Let $\mathbb{S} = \cup \mathbb{S}_n$ where the $\mathbb{S}_n$ are sets of order n symmetries. We can then define a strengthened energy by

$$E_{X,\mathbb{S},n}[\phi](\Sigma) = \sum_{i=0}^{n} \sum_{S\in\mathbb{S}} E_X[S\phi](\Sigma).$$

This notion of strengthening is due to Klainerman in some sense.

For surfaces Σ, Σ_i with causal, future directed normals, suppose we have a lens shaped region Ω (i.e. with top and bottom that are both causal surfaces, but no timelike side regions) with $\partial\Omega = \Sigma_i - \Sigma_1$. [I know this isn't quite correct subscripts and such, but I'm not quite sure what they are supposed to be.]

EG1: If X is a causal future-directed vector, then the dominant energy condition (or for null geodesics γ) implies that $E_X(\Sigma) \geq 0$.

EG2: (iii) (from the previous lecture, or for null geodesics γ), then $E_X(\Sigma_2) - E_X(\Sigma_1) = Bulk_x(\Omega)$

EG3: $E_{X,\mathbb{S},n}$ has the same properties.

Ex: The $\mathbb{R}^{3+1}$ wave equation. It has an energy related with ∂_t. If

$$T_n = \{\partial_1^{n_1}\partial_2^{n_2}\partial_3^{n_3} : \sum n_i = n\},$$

then $\|u(t)\|_{L^\infty}^2 = E_{\partial_t,T,1}[u](\{0\}\times\mathbb{R}^3)$.

This lets us see what problem is with problem is Kerr, which has no conserved energy quantity because ∂_t fails to be timelike everywhere. The normal vector to a $t=0$ slice of Kerr has a conserved energy, but then we don't get a Killing field. Thus you can only get positive non-conserved energy, or conserved energy that may be negative.

How is this overcome?

2. Morawetz estimate outside black holes

By this we mean an integrated local energy estimate, not something to do with the vector field $K = (t^2+r^2)\partial_t + 2rt\partial_r$, which she also did.

For the wave equation, $a=0$ (no spin, so Schwarzschild),

$$E_{\partial_t} \gtrsim \int_{exterior} \left(\left(\frac{\Delta}{t^2+a^2}\right)^2 |\partial_t|^2 + \left(\frac{r-3M}{r}\right)\frac{1}{r^2}(|\partial_t u|^2 + |\nabla_{s^2} u|/r^2)\right) r^2 dr d^2\omega_{s^2}.$$

We can think that as waves are moving out, $1/r^2$ goes like $1/t^2$, and so one can see why we can integrate over the exterior portion of the spacetime. One might worry about those orbiting null geodesics near the black hole, but they are at $3M$ and the term that vanishes there in the integral means they add nothing to this integral.

For $|a| << M$: The sphere of rotating orbits bifurcates, so $r-3M$ must be replaced by some measure of distance from the orbits.

Why is it useful? Application in $|a| << M$:

Let $T = \partial_t + \xi\frac{a}{r_+^2+a^2}\partial_\phi$ where $r_+ = M + \sqrt{M^2-a^2}$ and ξ is as in figure 1. This vector is timelike in the exterior region and becomes null at $r = r_+$. We have that $supp\nabla^{(a}T^{b)}$ is only where ξ has support, i.e. between $10M$ and $11M$. We then get $|\nabla^{(a}T^{b)}| \lesssim |a|/M$.

Let's assume we have a vector field A with a bulk of the form of the ugly integral a few paragraphs ago, and energy is bounded by the T energy. Then we'd say

$$\begin{aligned} E_T(t) - E_T(0) &= Bulk_T(\{t' \in [0,t]\}) \\ &\lesssim |a| Bulk_A \\ &\lesssim |a|(E_A(t) - E_A(0)) \\ &\lesssim |a|(E_T(t) + E_T(0)). \end{aligned}$$

This then gives that $E_T(t) \lesssim (1 + C|a|)E_T(0)$. This energy is not conserved, but it is positive, and can be uniformly bounded at later times by the original time.

For different cases and equations, a similar results was obtained by:

For $a = 0$, for the wave equation, By Blue and Soffer, By Blue-Sterbensz, Dafermos-Rodnianski, based on using an A introduced by Laba-Soffer.

For Maxwell: Blue, Sterbenz-Tataru (generic spherically symmetric black holes)

For Einstein: Holzegel (looks at something stronger than linearized gravity, and then proves decay estimates.)

For $|a| << M$: Done by Dafermos-Rodnianski (uniform bound on energy by looking at spectral decomposition, then found decay for certain frequencies), Tataru-Tohaneanu (spectral methods), Andersson-Blue (just using differential operators, no spectral methods)

For $|a| < M$: Dafer-R (got the estimates), Shlapentokh-Rothman

3. Killing tensors

A Killing p-tensor is a symmetric p tensor such that the symmetric part of the covariant derivative is equal to 0, i.e. $\nabla_{(b}K_{a_1,\cdots a_p)} = 0$.

With Killing vectors, we know a lot. If k is a Killing vector, then

(1) the flow generated by k is an isometry.
(2) the bulk term is equal to 0, so we have conserved energy for null geodesics and fields.
(3) L_X commutes with the wave, Maxwell and Weyl equations. This defines a symmetry of a PDE. Hence (assuming certain self adjointness properties), this allows for spectral decomposition that is compactable with the PDE we're looking at.

If K is a Killing tensor, we know much less.

(1) If γ is null geodesic then $\nabla_{\dot\gamma}(K_{a_1,\cdots a_n}\dot\gamma^{a_1}\cdots) = 0$
(2) If we are Ricci flat, then $[\nabla_a K^{ab}\nabla_b, \nabla^c\nabla_c]u = 0$, i.e. it commutes with the wave equation).

The relation between the two is less clear than we would like however...

For Kerr: Maxwell and linearized gravity can be decomposed into spinor or null frame, and some of those components have spectral decomposition (we can apply separation of variables).

4. Null geodesic model

Let

$$K^{ab} = \partial_\theta^a\partial_\theta^b + \frac{1}{\sin^2\theta}\partial_\phi^a\partial_\phi^b + 2a\partial_\phi^{(a}\partial_t^{b)} + a^2\sin^2\theta\partial_t^a\partial_t^b.$$

Also let $\Omega^2 = \frac{\Delta\Sigma}{(r^2+a^2)^2}$,$h = \frac{\Delta}{r^2+a^2}$ and $V_L = \frac{\Delta}{(r^2+a^2)^2} \to \frac{1}{r^2}(1-2M/r)$ as $a \to 0$. Let

$$\mathcal{R}^{ab} = -\partial_t^a\partial_t^b - \frac{4aM}{(r_+^2a^2)^2}\partial_t^{(a}\partial_\phi^{b)} - \frac{a^2}{(r^2+a^2)^2}\partial_\phi^a\partial_\phi^b + V_L K^{ab}.$$

We can then get $\Omega g^{ab} = h^2\partial_r^a\partial_r^b + \mathcal{R}^{ab}$, where g is the Kerr metric. Thus, we've managed to hide all the θ dependence in K and Ω.

Now let's look at null geodesic equations, $0 = \Omega^2 g^{ab}\dot\gamma_a\dot\gamma_b$. We can use the previous equality to get $(hg^{rr})^2\dot r^2 = -\mathcal{R}^{ab}\dot\gamma_a\dot\gamma_b \equiv -\mathcal{R}$. These are all Killing tensors contracted with $\dot\gamma$, and so this is a function of r alone. This was found in the 60s and 70s. $\dot r^2$ can never be negative, and so we have turning points when $\mathcal{R} = 0$. We have asymptotic orbits approaching from either side at maxes or mins of $\mathcal{R}$, and so there are orbits when $\mathcal{R} = 0$ and $\partial_r\mathcal{R} = 0$.

For $a = 0$, let the energy be $e = \partial_t^a\dot\gamma_a$, and the total angular momentum be $l^2 = K^{ab}\dot\gamma_a\dot\gamma_b$. l has all the angular terms so this makes some sense.

Consider orbit condition: Let $\partial_r(-e^2 + V_L l^2)$. e, l are constant and so we get zeros at $r = 3M$. In Kerr, it is much more complicated, but the condition is continuous in a. These orbits are unstable, which implies that $\partial_r^2\mathcal{R} < 0$. [Technically the other implication.]

Let $A = f\partial_r$, where f is not dependent on ϕ or θ. Then

$$2Bulk_A = \int \dot\gamma_b\dot\gamma_a L_a g^{ab} d\lambda = \int \dot\gamma_a\dot\gamma_b g^{ab}\Omega^2 L_A\Omega^{-2} + \dot\gamma_b\dot\gamma_a\Omega^{-2}L_A\Omega^2 g^{ab} d_\lambda.$$

The first term is zero, and so we just get the second one.

We calculate

$$\begin{aligned} L_A\Omega^2 g^{ab} &= (f\partial_r h^2 - 2h^2\partial_r f)\partial_r^a\partial_r^b + f\partial_r\mathcal{R}^{ab} \\ &= -2h^3(\partial_r(f/h))\partial_r^a\partial_r^b + f\partial_r\mathcal{R}^{ab}. \end{aligned}$$

Let $f = h\partial_r\mathcal{R}$ so that we get a perfect square for the last term. We then get

$$Bulk_A = \int h^4\Omega^2(-\partial_r^2\mathcal{R})(\dot\gamma_r)^2 + f(\partial_r\mathcal{R})^2 d\lambda.$$

The second term is manifestly nonnegative. The first term is positive by the unstable orbit condition above [which gives $\mathcal{R} < 0$].

For $a = 0$, we get $f = -h(2/r^3)(r - 3M)l^2$. This is bounded, and so $l^{-2}E_A \lesssim E_T$ and

$$Bulk_A \int A\Omega^{-2}(-\partial_r V_L)(\dot\gamma_R)^2 + (4/r^6)(r - 3M)^2 l^2 d\lambda.$$

We then get the Morawetz type estimate. I have as much of radial derivatives as I like, and I can trade in l^2 to get enough angular terms, and so I get it of the right form.

In Kerr, $\partial_r\mathcal{R}$ is a measure of distance from orbits, so we get what we said we had to do earlier.

For the other model equations?: We have here $L_A\Omega^2 g^{ab}\dot\gamma_a\dot\gamma_b \geq 0$. It is not the dominant energy condition that gives this, but some nice properties of geodesics. For the wave equation,

$$T_{ab} = \partial_a u\partial_b u - \frac{1}{2}g_{ab}\partial_a u\partial^a u$$

and we get $L_A\Omega^2 g^{ab}\partial_a u\partial_b \geq 0$ for the same reason as for null geodesics. There are then some tricks to get rid of the trace of T.

For Maxwell, $L_A\Omega^2 g^{ab}T_{ab}$ has no good properties. But, the middle component satisfies a wave-like equation, which implies Morawetz estimate for that component. Then, we can show control in the middle component implies control on the rest of the components.