Wang - Quasilocal mass

Fig 1

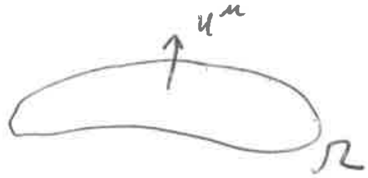

Fig 2

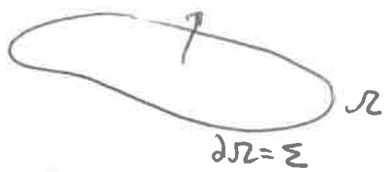

Fig 3

QUASI-LOCAL MASS IN GENERAL RELATIVITY

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Fundamental difficulty: There is no mass density for gravitation. This is implied by Einstein's equivalence principle. Therefore, an equation like

$$\text{mass} = \int_\Omega \text{mass density}$$

for Ω a spacelike region (which is called a bulk integral) is impossible.

Therefore, most of our understanding is on total mass of an isolated system, i.e. an asymptotically flat (AF) spacetime. The (total) mass is then measured as a flux 2-integral at asymptotic infinity.

It would be extremely useful to have a quasi-local description of mass. For instance, most physical models [such as for particles] are finitely extended regions.

In 1982, Penrose, for this reason, compiled a list of major unsolved problems in GR. The number 1 problem was: find a suitable definition of quasi-local mass and energy momentum.

Plan for the 2 days:

(1) Energy and mass in special relativity
(2) energy and mass in GR
(3) State the problem and expected properties (of quasi-local mass)
(4) Brief survey of known quasi-local mass constructions [end of day 1]
(5) Introduce proposal by Wang- S.T.Yau from 2009
(6) Applications of this new quasi-local mass, including invariant mass conjecture in GR and Quasi-local conserved quantities and dynamical formula for these alongside the Einstein equations

Consider a matter field in $\mathbb{R}^{3,1}$. Associated with it, there is an energy momentum tensor of matter density $T_{\mu\nu}$, a $(0,2)$ symmetric tensor, with $\partial^\mu T_{\mu\nu}$. Take $\Omega \subset \mathbb{R}^{3,1}$, a spacelike region. See figure 1. Let t^μ be a translating Killing field (unit) future directed. Let u^μ be the unit future time normal of Ω. Then we can define $\int_\Omega T_{\mu\nu}t^\mu u^\nu$ to be the energy intercepted by Ω and seen by the observer t^μ.

By the dominant energy condition (DEC), this integrand is non-negative. On the other hand, $T_{\mu\nu}t^\mu$ is divergence free. Thus it is dual to a closed 3 form, i.e. $d\omega$ for a 2-form ω (since we're in 3 dimensions). Thus we can write

$$\int_\Omega T_{\mu\nu}t^\mu u^\nu = \int_{\partial\Omega} \omega$$

for some ω that is linear in t^μ. We can minimize among all such t^μ, and that gives you a quasi-local mass. The value of the integral is the quasi-local mass.

The dual of this integral gives the quasi-local energy momentum 4-vector [since the integral is essentially just giving the time direction part of the full energy momentum 4-tensor].

Replacing t^μ by other Killing fields, we obtain (by plugging in a rotation Killing field) quasi-local angular momentum or (by plugging in a boost Killing field) quasi-local center of mass.

What about in GR? There is a $T_{\mu\nu}$ in GR, with $\nabla^\mu T_{\mu\nu}$. It appears in the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}.$$

Why can't we play the same game? $T_{\mu\nu}$ only accounts for energy contribution from matter fields. For example, Schwarzschild has $T_{\mu\nu} = 0$, but there is still gravitational mass. So we can't just integrate $T_{\mu\nu}$ to get a quasi-local mass. Plus, we might not have Killing fields to get conserved quantities anyway.

Recall that the Einstein equation is the Euler-Lagrange equation of the Einstein-Hilbert action. For a spacetime domain, this [integrated] action is

$$\frac{1}{16\pi} \int_M R + \frac{1}{8\pi} \int_{\partial M} K + \int_M L(g, \phi)$$

where the last integrand is the Lagrangian for the matter fields. The K is really the mean curvature. That entire integral with K is a divergence term, which we can rewrite as $\int_M \partial_\mu I^\mu$. Thus w get some quadratic term in the first derivative of the metric.

We can apply Hamilton-Jacobi analysis to this. We obtain $T^*_{\mu\nu}$ (a quadratic expression of 1st derivatives of the metric), the Einstein pseudo tensor. We still have $\partial^\mu T^*_{\mu\nu} = 0$. However, it is not symmetric, and not even a tensor! It depends on the coordinate system we pick. In general, mass density is such a quadratic expression of 1st derivatives, but locally we can make 1st derivatives of the metric 0, so this can't quite do what we want correctly.

However, in an AF spacetime, we can choose coordinates, and express mass as a 2-integral at the infinite boundary.

Total mass definitions: Spatial infinity (or ADM) energy momentum. Take initial data (M, g_{ij}, P_{ij}), where P is the 2nd fundament form, and g is the induced metric which is AF. We say that $(M, g, P$ is asymptotically flat if there exists a compact set $K \subset M$ such that $M \setminus K$ is a union of ends (i.e. diffeomorphic to complements of balls in $\mathbb{R}^3$), and $g_{ij} - \delta_{ij} = o_2(r^{-\alpha})$ and $P_{ij} = o_1(r^{-\alpha-1})$ for $\alpha > \frac{1}{2}$. Without that last condition on α, there are some problems that arise. We can then define ADM energy by

$$E = \lim_{r\to\infty} \frac{1}{16\pi} \int_{\Sigma_r} (g_{ij,j} - g_{jj,i}) \nu^i d\Sigma_r$$

for the coordinate sphere Σ_r. Similarly, we can define ADM momentum by

$$P_i = \lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \pi_{ij}\nu^j d\Sigma_r$$

where $\pi_{ij} = P_{ij} - \mathrm{tr}_g P g_{ij}$. In this case we have the positive energy theorem, which was proven by Schoen, Yau and Witten. Under the DEC, we have that (E, P_i) is future non-spacelike, with $m = \sqrt{E^2 - \sum P_i^2} \geq 0$, and this quantity equals zero only if the data corresponds to a hypersurface in $\mathbb{R}^{3,1}$.

Null infinity (Bondi-Sachs) energy momentum. Suppose that outside a compact set, there exists a spacetime coordinate system, W (retarded time), r, x^1 and x^2 such that the spacetime metric is of the form

$$-UVdw^2 - 2Udwdr + g_{ab}(dx^a + u^a dw)(dx^b + u^b dw)$$

We can write down the expansion for the metric coefficients, and get

$$V = 1 - \frac{2m(w, x^a)}{r} + \cdots .$$

where this expansion defines $m(w, x^a)$, which is called the mass aspect function. On each hypersurface $w = c$, $E = \int_{S^2(\infty)} 2m(c, x^a)dS^2$ and $P_i = \int_{S^2(\infty)} 2m(c, x^a)\tilde{x}^i dS^2$ where $\tilde{x}^i$ are the first 3 eigenfunctions of S^2. There is also a PMT for this.

We can see that these two total masses are both expressed as flux 2-integrals at infinity, and this is possible because gravitation is weak at infinity, which says that there exists asymptotic flat coordinates. The problem is that there is no ground state for gravity.

But what if gravitation is strong?

Problem of quasi-local mass: For any spacelike 2-surface in a spacetime that bounds a spacelike region (see fig 2), we want to define quasi-local mass and quasi-local energy-momentum. The expected properties are

(1) Positivity (at least for "large convex" spheres, since gravitational binding energy may be negative locally)
(2) Rigidity. every surface in $\mathbb{R}^3$ has mass equal to 0.
(3) Asymptotics. The large sphere limit should be ADM mass (or Bondi mass on null surfaces.) And the small sphere limit should recover the energy-momentum tensor if there is matter, of the Bel-Robinson tensor if it is vacuum.
(4) Conservation law or monotonicity. For example, in a spacelike direction we should be able to control rate of change (applications in the Penrose inequality), and similarly in causal directions (applications in the Einstein equations)

So far, there are 4 [main] different approaches to quasi-local mass [for AF spacetimes].

(1) Variational approach (Bartnik, Bray, others). Based on a quasi-localization of the ADM mass.

(2) Twistor or spinor approach (Penrose, Dougan-Mason, Ludvigsen-Vickers, others).
(3) Hawking mass (Hawking, Geroch, others).
(4) Hamilton-Jacobi method (Brown-York, Hawking-Horowitz, Kijawski, Liu-Yau, others)

We will focus on the last two approaches, since they seem to have more application to mathematical relativity and they are closely related to each other.

Both of them use only the induced metric on the (spacelike) surface Σ and the mean curvature vector. Recall that on any spacelike 2-surface, there exists a unique normal vector field $\vec{H}$ such that the variation

$$\delta_V|\Sigma| = -\int_\Sigma \langle H, V\rangle d\Sigma$$

for any vector field V along Σ. See figure 3. In particular, when V is a null normal, $\delta_V|\Sigma|$ is the null expansion, $[\theta^+]$. Thus $\vec{H}$ is closely related to energy or mass by the Penrose singularity theorem. If we assume H is spacelike, then

$$\sqrt{-8\rho\mu} = \sqrt{-8\mathrm{tr}\xi\mathrm{tr}\underline{\xi}} = |H| > 0.$$

Hawking mass:

$$m_H = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma |\vec{H}|^2 d\Sigma\right)$$

There is also a time symmetric version ($P = 0$). Replace $|\vec{H}|^2$ by k^2 where k is the mean curvature off Σ as the boundary of Ω. In general we have $|\vec{H}|^2 = k^2 - (\mathrm{tr}_\Sigma P)^2$, which is why we can do this.

For a spherically symmetric space time the metric can be written

$$g_{ab}dx^a dx^b + r^2(x^1, x^2)d\Omega^2,$$

where the first term is on a Lorentz manifold $Q^{1,1}$. We have $\vec{H} = \frac{-2}{r}\nabla r$. Then

$$m_H = \frac{r}{2}(1 - |\nabla r|^2)$$

We can compute

$$\partial_c m_H = -4\pi r^2 \partial^a r(T^a_c - \delta^a_c \mathrm{tr}gT)$$

for $T_{ab}dx^a dx^b + r^2 S d\Omega^2$. Using the DEC, we can then determine in which direction this mass can be increased.

This mass also has nice monotonicity along inverse mean curvature flow in the time symmetric case. For this flow, we say $\frac{\partial\Sigma}{\partial t} = \frac{1}{k}\nu$. This is instrumental in the proof of the Riemannian Penrose inequality of Huisken-Ilmanen. They show that the ADM mass is greater than or equal to $\sqrt{\frac{|\Sigma|}{16\pi}}$, where Σ is the outermost minimal surface. This is because m_H is increasing and it approaches the ADM mass.

Such monotonicity holds for spacetime IMCF (Freidrich) (he used it for the non-time-symmetric Penrose inequality) and for IMCF on null hypersurfaces. In that last case, there is also a Penrose inequality for the Bondi mass, which is again greater than or equal to same quantity.

The existence of the flow and the asymptotic behavior of Hawking mass remains obstacles to the proof.

Hamilton-Jacobi method:

$$M = \frac{1}{8\pi}\int_\Sigma H_0 \, d\Sigma - \frac{1}{8\pi}\int_\Sigma |\vec{H}| \, d\Sigma$$

where H_0 is the mean curvature of isometric embedding of Σ into $\mathbb{R}^4$. (If $K_\Sigma > 0$ then we have uniqueness.) Notice that here we have a lower power of H. This form of the mass is due to Liu-Yau, Kijowski, Man, etc.

In the time symmetric case, we can replace $|\vec{H}|$ with k. This is the Brown-York mass, which has good positivity by Shi-Tam.

In the spherically symmetric case, we have $M = r(1 - |\nabla r|)$, which implies that $m_H = M - M^2/2r$ at any $p \in Q$, where Q again, is the quotient manifold of the spacetime by factoring out the S^2.

Rigidity: We test any $\Sigma \in \mathbb{R}^{3,1}$. If $\Sigma \in \mathbb{R}^3$, then $M = 0$. But $m_H < 0$ unless Σ is a round sphere! If $\Sigma \in C$, a light cone in $\mathbb{R}^{3,1}$, then $m_H = 0$, but $M > 0$ unless Σ is again a round sphere.

We can compare the masses on the Schwarzschild spacetime also. If on totally geodesic slice, then both are good, but if we slice a light cone or a round cone, then both are greater than the Schwarzschild mass and can be made arbitrarily large...

There exists a slice in Schwarzschild, $-(1-2m/r)dt^2+(1-2m/r)^{-1}dr^2+r^2d\Omega^2$, which is essentially Minkowskian. Consider a hypersurface $t = f(r)$ with $f'(r) = \sqrt{2mr}/(1-2m/r)$. The induced metric is exactly the same as on $\mathbb{R}^3$ and the 2nd fundamental for it is equal to $\sqrt{2m}r^{-3/2}(-\frac{1}{2}dr^2 + r^2 d\Omega^2)$. Thus the ADM mass for this slice is zero!