Bieri - Null hypersurfaces

Fig 1

Fig 2

Fig 3

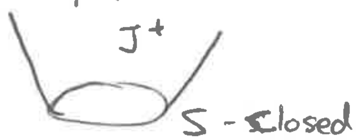


Fig 4

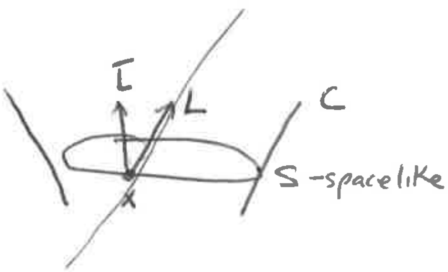


Fig 5

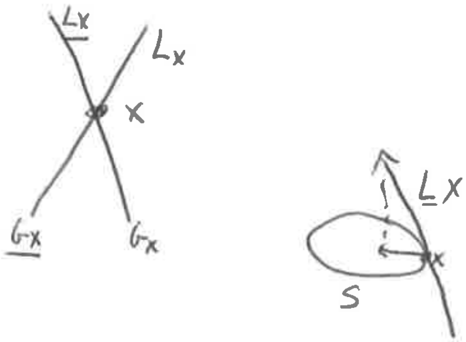


Fig 6

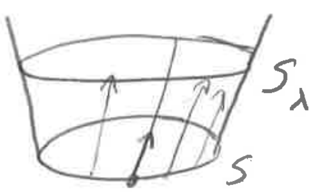


Fig 7

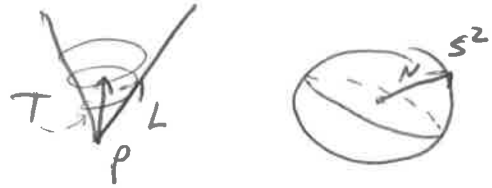


Fig 8

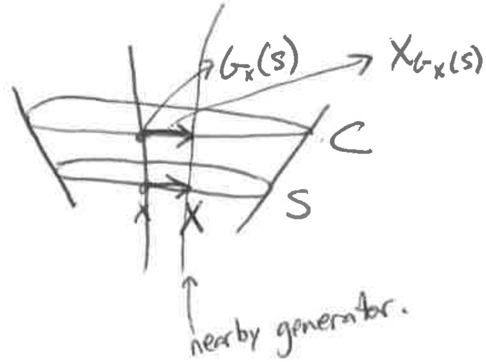
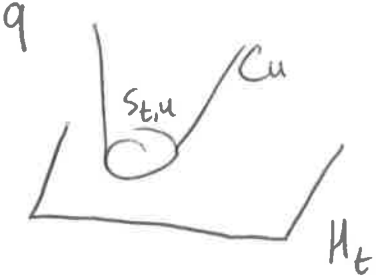


Fig 9



NULL HYPERSURFACES IN LORENTZIAN SPACETIMES

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Outline:

(1) Lorentzian spacetimes
(2) Foliations
(3) Null Hypersurfaces
(4) Their Role in GR

We will consider spacetimes (M^4, g), where M is a 4-dimensional manifold, and g is a Lorentzian metric solving the Einstein equations (1),

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

where T is the energy-momentum tensor.

If we set $T = 0$, we still get interesting behavior. We can then derive the Einstein vacuum equations, $R_{\mu\nu} = 0$, (2). If we do not say otherwise, we will assume that our spacetime is vacuum.

Assume (M, g) is oriented, as smooth as we need and solves (2). We look at asymptotically flat (AF) solutions, and we will care about things like neutron stars merging and releasing gravitational radiation. See fig 1: it releases radiation on null hypersurfaces coming off the binary inspiral.

We will use the idea of causal curves. If we want to send information, we can use timelike or null curves. This notion of causality is important. Hermann Weyl first introduced it, though he though of things differently than we do now.

We say γ is a causal curve in M if it is a differentiable curve where the tangent vector $\dot{\gamma}$ is either timelike or null at each point $p \in M$.

See figure 2. The causal future of a point $p \in M$, $J^+(p)$, is the set of all points $q \in M$ for which there exists a future directed causal curve from p to q. Also, $J^+(S) = \{q \in M : q \in J^+(p) \text{ for some } p \in S\}$. The past, J^-, is defined similarly.

The boundaries $\partial J^+(S)$ and $\partial J^-(S)$ for closed sets S are null hypersurfaces. They are generated by null geodesic segments.

These null hypersurfaces are realized as level sets of a function u which satisfies the eikonal equation, $g^{\mu\nu}\partial_\mu u \partial_\nu u$.

In general, we have the following definition:

Definition 0.1. Some hypersurface H is called a <u>null hypersurface</u> if at each point $x \in H$ the induced metric is degenerate, i.e. there exists some $L \neq 0$ lying in $T_x H$ such that $g_x(L, X) = 0$ for any $X \in T_x H$. [Thus, there is a tangent vector to the hypersurface such that everything else in the tangent space is "co-null" with it.]

The hyperplane $T_xH \subset T_xM$ can also be defined by a covector $\xi \in T_x^*M$. Define $T_xH = \{X \in T_xM : \xi \cdot X = 0\}$. Thus we can represent H as a (0-)level set of a function u. In that case, $\xi = du(x)$. We can set, therefore, in an arbitrary frame, $L^\mu = -g^{\mu\nu}\partial_\nu u$. Then we have $g(L,X) = -du \cdot X$ and L is g orthogonal to H. Thus H is a null hypersurface if and only if $L_x \in T_xH$ for every point $x \in H$.

Each of the level sets of u will be a null hypersurface. If we take $X = L$, since $g(L,X) = 0$, we then have $g(L,L) = 0$ and so $g_{\mu\nu}L^\mu L^\nu = 0$. In terms of du, this implies the eikonal equation again, $g^{\mu\nu}\partial_\mu u \partial_\nu u$.

Important: L is actually a geodesic vector field. Proof: easy.

See fig 4. We take a null cone C and a cross section S. How could we construct such a C from S? The integral lines through L are null curves, which we call G_x. Look at $P_x = (T_xS)^\perp \subset T_xM$. P_x is a 2-dimensional linear space. (see fig 5) This lets me differentiate in outward and inward directions. Select a future directed null vector L_x which generates a null line G_x for every $x \in S$. This is defined up to a positive constant and gives a vector field L on S. We can transform it like $L \mapsto aL$, where a is a positive function on S.

We can then have $g(L,\underline{L}) = -2$, where we have the transformation $\underline{L} \mapsto \frac{1}{a}\underline{L}$. We can have initial conditions on S, and then $C = \cup_{x\in S}G_x$ (see figure 6).

Define $\chi(x,y) = g(\nabla_x L, Y)$ for $X, Y \in T_xS$ to be the 2nd fund form of S with respect to C. It is a symmetric bilinear form living on S.

Let $S_\lambda = \{G_x(\lambda) : x \in S\}$, which is something like the λ level set of the geodesics. Define an affine function s on C by requiring $Ls = 1$ and $s|_{S_0} = 0$.

If C is the light cone of a point, the situation is pretty simple. Start with $p \in M$. See figure 7. Pick at p a unique future directed timelike vector T. Consider a spacelike hypersurface orthogonal to T. In here, look at the unit sphere S^2, and take unit normals N. For every unit spacelike vector $N \in S^2$ we have $L = T + N$. Now, L is future directed null at p.

Define a flow ϕ_t which is generated by L on C: $\phi_t(G_x(s)) = G_x(s+t)$. We can extend a vector field X on S (tangential to S) to C by taking $[L,X] = 0$. Then X is tangent to each level set S_λ. See fig 8. We then have $X_{G_x(s)} = \phi_s X_x$. This is a Jacobi field of sorts.

It is sufficient to look at a vector $X_x \in T_xS$, transport it and then use Jacobi fields.

Once we have our nice Jacobi fields, we can derive variational formulas. We can write the 1st variational formula. If we take Jacobi fields X and Y along a given generator, and alow γ to be the induced metric on S, i.e. $\gamma = g|_{TS}$, then $\partial_s\gamma(X,Y) = 2\chi(X,Y)$. [Thus this 2nd fundamental form is the derivative of the induced metric as we change s, i.e. as we go up H.]

Why should we care? For tomorrow: We can add the other second fundamental form, $\underline{\chi}(X,Y) = g(\nabla_X L, Y)$, both of which we can split into trace ($\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$) and traceless parts ($\hat{\chi}$, $\underline{\hat{chi}}$). The traceless parts we will call shears. [

We also need the torsion: $\zeta(X) = \frac{1}{2}g(\nabla_X L, \underline{L})$ for any $X \in TS$. We can then show that $\nabla_L \underline{L} = -2Z$ defines the corresponding vector field Z.

These shears will be a main ingredient tomorrow.

A closed trapped surface S is a 2-dimensional surface such that $\mathrm{tr}\chi < 0$ on it (since this means that the outward nulls are going in). The Penrose incompleteness theorem says, with some other assumptions, that there are future null incomplete geodesics.

Christodoulou in 2008 showed the formation of a BH in the vacuum case. There were later refinements later by Klainerman and Rodnianski. His work shows that if there is enough energy coming in by gravitational waves, we can form a BH! We will talk about this radiation tomorrow.

Gravitational Radiation goes out on null geodesics, and we usually assume that we are sitting out at null infinity (i.e. at $t \to \infty$). Some of the equations we've talked about have nice limits at null infinity. For instance, $\lim_{Cu,t\to\infty} r\mathrm{tr}\chi = H_1$, some limit. Similarly $\underline{\chi}$ will have limits, and correspondingly for $\lim r^2\hat{\chi} = H_3$ and $\lim r\underline{\hat{\chi}} = H_4$. [I'm not sure if these fall offs are correct, but they are what she wrote.]

Using L and $\underline{L}$, we can decompose curvature using frame $\{e_1, e_2, \underline{L}, L\}$ (which is a completed orthonormal frame), and we'll get curvature decay for some pieces.

On such a surface S, by Codazzi,

$$\mathrm{div}_S\hat{\chi} = -\hat{\chi}\cdot\zeta_{\frac{1}{2}}(\nabla_S \mathrm{tr}\chi + \zeta\mathrm{tr}\chi) - \text{“}\mathit{curvature}\text{”} - 8\pi T_{AL}.$$

There is a corresponding equation for $\underline{\hat{\chi}}$. We also have a propagation equation for $\mathrm{tr}\chi$,

$$\partial_s \mathrm{tr}\underline{\chi} = -\frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - 2\underline{\mu} + 2|\zeta|^2$$

where $\underline{\mu}$ is the mass aspect function.

$$\partial\mathrm{tr}\chi = -\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2 + 8\pi T_{ii}$$

We then can write $\partial_s\nabla_S\mathrm{tr}\chi$. We have

$$\underline{\mu} = K + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + \mathrm{div}_S\zeta.$$

The Hawking mass is then

$$m(t,u) = \frac{r}{2}\left(1 + \frac{1}{16\pi}\int_{S_{t,u}} \mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right).$$

The Gauss-Bonnet theorem then gives

$$\int_{S_{t,u}} \underline{\mu} = 4\pi \left(1 + \frac{1}{16\pi} \int_{S_{t,u}} \mathrm{tr}\chi \mathrm{tr}\underline{\chi} \right) = \frac{8\pi}{r} m.$$

Thus we have $m \to M$, the Bondi mass, as $t \to \infty$, i.e. at null infinity. Also, $\partial M / \partial u$ is something interesting too.

In general we don't have the differentiability needed to make all these things pretty.