Huang- Density Theorems

Fig 1

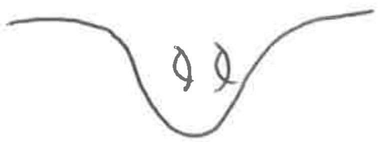
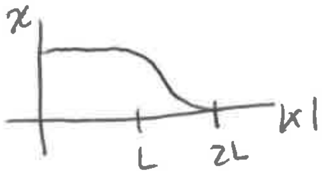


Fig 2



DENSITY THEOREMS FOR THE EINSTEIN CONSTRAINT EQUATIONS

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We will discuss perturbation results of initial data sets in the set of initial data sets. We will discuss global perturbations.

We start with the vacuum, time-symmetric case. In this case, the constraints reduce to $R_g = 0$, where we are have (M^3, g) with g asymptotically flat (AF) i.e. outside a compact set, we have $g_{ij} = \delta_{ij} + o(|s|^{-1})$ (see fig 1).

We want to simplify the asymptotics. Take a cutoff χ as in figure 2. Let $\hat{g} = \chi g + (1-\chi)\delta$. For this metric, we have $R_{\hat{g}} = 0$ on the center and the asymptotic region, but it is not in the annulus region. Assume $|\nabla\chi| =)(L^{-1})$ and $|\nabla^2\chi| = O(L^{-2})$. For this, we have that $R_{\hat{g}} = O(L^{-3})$. We can't make it exactly 0 though, since it would violate the PMT.

However, let $\bar{g} = u^4\hat{g}$ with $u \to 1$ as $|x| \to \infty$. We then get

$$R_{\bar{g}} = -\frac{(\Delta_{\hat{g}}u - \frac{1}{8}R_{\hat{g}}u)}{\frac{1}{8}u^5} = 0$$

Thus we need to solve $\Delta_{\hat{g}}u - \frac{1}{8}R_{\hat{g}}u = 0$. This operator is actually Fredholm in some appropriate weighted space, and it has no kernel. That gives existence and uniqueness of the solution u.

Note that outside a compact set, u is just a harmonic function, and so we can write it as an expansion

$$u = 1 + A/|x| + \vec{B}\cdot x/|x|^3 + D/|x|^2 + \cdots .$$

This metric is very much like the Schwarzschild metric. This observation is due to Schoen-Yau. If the cutoff region is large enough, we can show that $\bar{g} \to g$ as $L \to \infty$ and also that $\bar{E} \to E$ as $L \to \infty$.

We can see if the scalar curvature is positive or negative depending on A; it is encoded there, with the (ADM) energy.

To summarize, we have shown that conformally flat scalar flat metrics are dense among AF scalar flat metrics.

For the vacuum, non-time symmetric case we take (M^3, g, k) with equations $R_g - |k|_g^2 + (\mathrm{tr}_g k)^2 = 2\mu$ and $\mathrm{div}_g(k - (\mathrm{tr}_g k)g) = J$. Our vacuum assumption gives us that $\mu = |J| = 0$. We also assume that $g = \delta + o(|x|^{-1})$ and $k = o(|x|^{-2})$. Introduce $\pi = k - (\mathrm{tr}_g k)g$. We can then rewrite the constraints as

$$\Phi(g,\pi) := \left(R_g - |\pi|_g^2 + \frac{1}{2}(\mathrm{tr}_g\pi)^2, \mathrm{div}_g(\pi)\right) = (0,0).$$

Let's try the same idea by cutting off the asymptotic region. Let $\hat{g} = \chi g + (1-\chi)\delta$ and $\hat{\pi} = \chi\pi$ (since momentum is zero for a flat slice).

We first compute that $\Phi(\hat{g}, \hat{\pi}) = (0, 0)$ on B_L and on $M \setminus B_{2L}$. On the annulus it is $(o(L^{-3}), o(L^{-3}))$. We again can't solve this locally on the annulus.

Corvino-Schoen in 2005 introduced a scalar function $u \to 1$ like a conformal factor alongside a vector function $\vec{X} \to 0$. Let $\bar{g} = u^4 g$ and $\bar{\pi} = u^2(\hat{\pi} + \mathcal{L}_{\hat{g}} X)$ where $\mathcal{L}_{\hat{g}} X = L_X \hat{g} - (\mathrm{div}_{\hat{g}} X)\hat{g}$. [This is almost, but not quite, the conformal Killing operator.] We plug these into the constraint equation and try to solve $\Phi(\bar{g}, \bar{\pi}) = (0, 0)$ for u and X. We get

$$0 = \Delta_{\hat{g}} u - \frac{1}{8}(R_{\hat{g}} u - |\hat{\pi}|^2_{\hat{g}} + (\mathrm{tr}_{\hat{g}}\pi)^2)u + DX * \hat{\pi} + DX * DX$$

$$0 = \Delta_{\hat{g}} X_i + \mathrm{div}_{\hat{g}}\hat{\pi} + Du * \hat{\pi} + Du * DX$$

This system is elliptic! Also, the second terms in each equation have nice decay. So again it is Fredholm over some appropriate spaces. However, it's not clear this map is surjective, even in the linearized version.

They found $u \to 1$ and $X \to 0$ and compactly supported perturbation symmetric $(0, 2)$-tensors h and ω such that

$$\Phi(u^4\hat{g} + h, u^2(\hat{\pi} + \mathcal{L}_{\hat{g}} X) + \omega) = (0, 0).$$

This still has very nice asymptotics since the perturbations are just on a compact set.

As $L \to \infty$, this data converges to the original metric (i.e. the perturbations also go to zero), and $\bar{E} \to E$ and $\bar{P} \to P$.

We get $\Delta_\delta u = o(|x|^{-3})$ and $\Delta_\delta X_i = o(|x|^{-3})$, and so u, X_i are harmonic up to higher order, i.e. $u = 1 + A/|x| + o(x^{-1})$ and $X_i = B_i/|x| + o(|x|^{-1})$.

Definition 0.1. We say that if AF (g, π) satisfies $g = g^4\delta$ and $\pi = u^2(\mathcal{L}_\delta X)$ outside a compact set then they are AF with harmonic asymptotics.

To summarize their result,

Proposition 0.2. *Vacuum AF data sets with harmonic asymptotics are dense among vacuum data sets.*

Applications: Let C be the center of mass. It is defined by

$$C^\alpha = \frac{1}{16\pi}\int_{|x|=\infty}\left[x^\alpha\left(g_{ij,i} - g_{ii,j}\right)\frac{x_j}{|x|} - \left(g_{i\alpha}\frac{x^i}{|x|} - g_{ii}\frac{x^\alpha}{|x|}\right)\right]d\sigma_0$$

It is not clear that this should even converge since we are one order higher in x. Given a rotation vector field X (e.g. $X = x_1\partial_2 - x_2\partial_1$), we define the angular momentum with respect to X to be

$$J(X) = \frac{1}{8\pi}\int_{|x|=\infty}\pi_{ij}X^i\frac{x^j}{|x|}d\sigma_0$$

Again, for the same reasons, it is not clear this integral should converge. In general it doesn't!

Definition 0.3. (g,π) satisfies the Regge-Teitelboim condition if $g(x)$ is "asymptotically even," (i.e. it looks like $(1+A/|x|+\cdots)\delta)$ and $\pi(x)$ is "asymptotically odd" (i.e. roughly it looks like $(B\cdot x/|x|^3+\cdots)$).

The center of mass and angular momentum are well-defined under the R-T condition.

Let's look back at the Corvino-Schoen method. If we write $\hat{g}=g+(1-\chi)(\delta-g)$, where $\delta-g=o(1/|x|)$ and $\hat{\pi}=\pi+(1-\chi)\pi$, where the last piece is $o(|x|^{-2})$.

We can find σ,τ, small perturbations, then construct $\hat{g}=g+\sigma$, $\hat{\pi}=\pi+\tau$ and then run through the same construction as before. We assume the perturbations σ,τ satisfy the linearized Einstein equations, $\sum_{ij}\sigma_{ij,ij}-\sigma_{ii,jj}=0$ and $\sum_i \pi_{ij,i}=0$.

Theorem 0.4 (H.-Schoen - M.T.Wang)**.** *Given (g,π) vacuum and given $\vec{\alpha}\in\mathbb{R}^3$, $\vec{\gamma}\in\mathbb{R}^3$, we can construct nearby data $(\bar{g},\bar{\pi})$ such that $\bar{E}=E$, $\bar{P}=P$ and $\bar{C}=C+\vec{\alpha}$ and $\bar{J}(X)=J(X)+\vec{\gamma}$.*

Theorem 0.5 (H. - Corvino)**.** *Given (g,π), then exists $(\bar{g},\bar{\pi})$ close to (g,π) and $\bar{E}\approx E$, $\bar{P}\approx P$, but $\bar{C}=\infty$ and $\bar{J}=\pi$.*

Thus bad slices and good slices are dense in the overall set!

What about the non-vacuum case? Given (g,π) which satisfies $\mu\geq|J|_g$ (J is not angular momentum, is from dominant energy condition (DEC)), can we perturb by cut-ff techniques and still have it satisfy the DEC?

Theorem 0.6 (Eichmair- H. - D. Lee - Schoen)**.** *Given (g,π) with $\mu\geq|J|_g$ and $\epsilon>0$, then there exists $(\bar{g},\bar{\pi})$ which is ϵ close to (g,π) and a positive constant $\gamma>0$ such that $\bar{\mu}\geq(1+\gamma)|\bar{J}|_{\bar{J}}$.*

Once you have this gap, you can do all the cut-off tricks we did before to get the same construction to work here too.