Wong - Quasi-local mass 2

Fig 1

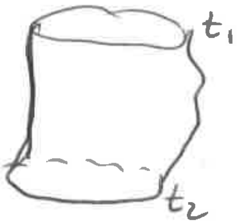


Fig 2

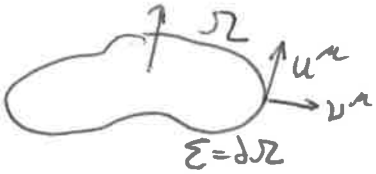


Fig 3

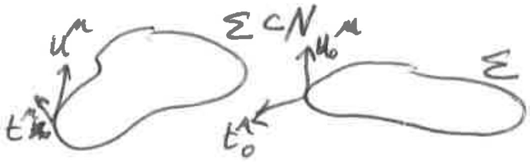


Fig 4

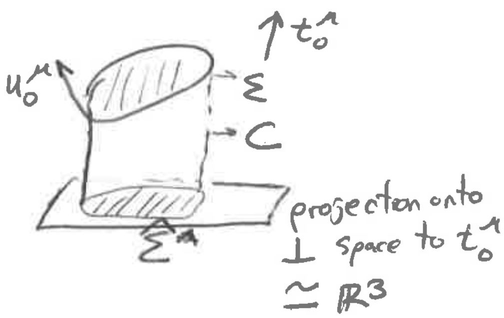


Fig 5

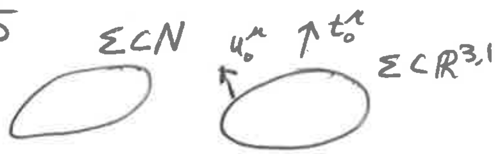
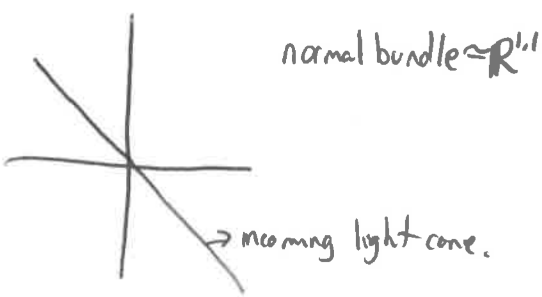


Fig 6



QUASI-LOCAL MASS IN GENERAL RELATIVITY

MU-TAO WANG

Today, I want to focus on the rigidity property. This is based on the belief that a physically valid definition of mass should vanish on Minkowski data. It turns out that this property is difficult to satisfy.

Rigidity statement of positivity of ADM mass: Take (M, g_{ij}, P_{ij}) with $g_{ij}-\delta_{ij} = o_2(r^{-\alpha})$ and $P_{ij} = o_1(r^{-\alpha-1})$ for $\alpha > 1/2$. Then the ADM mass is ≥ 0 and it is equal to 0 if and only if (M, g_{ij}, P_{ij}) is Minkowski data.

Let the Hawking mass be

$$m_H = \sqrt{\frac{|\Sigma|}{16\pi}}(1 - \frac{1}{16\pi}\int_\Sigma |\vec{H}|^2 d\Sigma)$$

and the Bondi mass be

$$M_B = \frac{1}{8\pi}\int_\Sigma H_0 \, d\Sigma - \frac{1}{8\pi}\int_\Sigma |\vec{H}| d\Sigma.$$

Rigidity: [Rigidity means that the quasi-local mass of a slice in Minkowski space should be zero.] We consider any [spacelike] $\Sigma \in \mathbb{R}^{3,1}$. We have that if $\Sigma \in \mathbb{R}^3$, then the Bondi mass is zero, $M_B = 0$. However, m_H is negative unless Σ is a round sphere! If $\Sigma \in C$, a light cone in $\mathbb{R}^{3,1}$, then $m_H = 0$, but $M > 0$ unless Σ is again a round sphere.

Recall that H_0 is the mean curvature of the isometric embedding of Σ into $\mathbb{R}^3$, i.e. we are taking $\mathbb{R}^3$ as our reference space and comparing the two. To anchor the rigidity property, it is thus natural to consider $\mathbb{R}^{3,1}$ as the reference space.

We will revisit the Hamilton-Jacobi method of deriving quasi-local mass. By Brown-York, Hawking-Horowitz: Apply Hamilton-Jacobi analysis of the Einstein-Hilbert functional to the time history of a bounded spacelike region. See figure 1. We obtain a surface Hamiltonian as a 2-integral on the terminal surface.

The surface Hamiltonian is: Let Σ be a spacelike 2-surface in a spacetime N. Suppose $\Sigma = \partial\Omega$, where Ω is a spacelike region. See figure 2. Take u^μ to be the future timelike unit normal of Ω and v^μ to be the outward spacelike unit normal of $\Sigma = \partial\Omega$. Let t^μ be a future timelike unit vector field along Σ. This plays the role of an observer. We can decompose $t^\mu = Nu^\mu + N^\mu$ where N is the lapse and N^μ is the shift vector. Then the Hamiltonian is

$$\mathcal{H}(t^\mu, u^\mu) = -\frac{1}{8\pi}\int_\Sigma (Nk - N^\mu \pi_{\mu\nu} v^\nu) d\Sigma.$$

Here, k is the mean curvature of Σ as $\partial\Omega$ in the direction of v^μ. Also, $\pi_{\mu\nu}$ is the conjugate momentum of Ω. In fact, the definition does not depend on Ω, but only on the frame $\{u^\mu, v^\mu\}$ along with Σ.

Recipe: Choose a reference space and a reference isometric embedding. See fig 3. Then the quasi-local energy is just defined to be the physical surface Hamiltonian $\mathcal{H}(t^\mu, u^\mu)$ minus the reference surface Hamiltonian $\mathcal{H}(t_0^\mu, u_0^\mu)$.

Question: How do we choose the reference space and (t_0^μ, u_0^μ) along Σ in the reference space?

Brown-York and L-Y both used $\mathbb{R}^3$ as the reference space. Brown-York's choice is gauge dependent, so we need to choose a particular u^μ. We then take $t^\mu = u^\mu$. This leads to the Brown-York mass. For L-Y, they chose u^μ to be the future timelike direction that is orthogonal to $\vec{H}$. They call this the binormal direction. We then take $t^\mu = u^\mu$ again.

Suppose we fix an isometric embedding $X : \Sigma \to \mathbb{R}^{3,1}$. Let t_0^μ be a future timelike unit translating Killing field. Take u_0^μ to be in the direction of the normal part of t_0^μ. Consider

$$\mathcal{H}(t_0^\mu, u_0^\mu) = \frac{1}{8\pi}\int_{\hat{\Sigma}} \hat{k} d\hat{\Sigma}.$$

See figure 4. Here $\hat{k}$ is the mean curvature of $\hat{\Sigma}$. This is a conservation law.

Suppose N^μ, the shift vector, is tangent to Σ. This implies that t_0^μ [is tangent to? perpendicular?] to the timelike hypersurface C. Thus

$$\mathcal{H}(t_0^\mu, u_0^\mu) = \frac{1}{8\pi}\int_{\Sigma} \pi^C_{\mu\nu} t_0^\mu u_0^\nu$$

where $\pi^C_{\mu\nu}$ is the conjugate momentum of C. We have $\nabla^\nu(\pi^C_{\mu\nu} t_0^\mu) = 0$ on C. This is how we get the previous paragraph's formula for $\mathcal{H}$.

Proposal of W-Yau (2009): Take $\Sigma \subset N$ in a physical spacetime. Suppose an isometric embedding $X : \Sigma \in\hookrightarrow \mathbb{R}^{3,1}$ and a t_0^μ, a translating Killing field are given. See figure 5. We need a (t^μ, u^μ) along the physical surface to define the physical surface Hamiltonian.

Canonical gauge: We claim that if $\vec{H}$ of $\Sigma \subset N$ is spacelike, then there exits a unique (t^μ, u^μ) such that

(1) The expansion of $\Sigma \subset N$ along t^μ equals the expansion of $\Sigma \subset \mathbb{R}^{3,1}$ along t_0^μ, i.e. the area change along both t^μ and t_0^μ are the same.
(2) The lapse and shift vector of t^μ and t_0^μ along u^μ and u_0^μ, respectively, are the same.

Define $E(\Sigma, X, t_0^\mu) := \mathcal{H}(t^\mu, u^\mu) - \mathcal{H}(t_0^\mu, u_0^\mu)$, where $\Sigma \subset N$.

Both m_H and M_B can be expressed in terms of the induced metric and the mean curvature vector $|\vec{H}|$. This new expression can be expressed in terms of the induced metric, $\vec{H}$ and $\tau = -\eta_{\mu\nu} X^\mu t_0^\nu$, where η is the flat metric and X is the position vector of the embedding. But the direction of $\vec{H}$ is also used. There is a connection one form, α_H, that is determined by the mean curvature vector. $\vec{H}$ is a normal vector. Consider the normal bundle in Figure 6. If reflect $\vec{H}$ along the incoming light cone we get $\vec{J}$. If $\vec{H}$ is inward spacelike, then $\vec{J}$ is future timelike. Then $\alpha_H(V) = \langle \nabla_V \frac{\vec{J}}{|\vec{H}|}, \frac{\vec{H}}{|\vec{H}|} \rangle$, which is the torsion in the binormal direction.

We can prove the positivity of $E(\Sigma, X, t_0^\mu)$ when $\vec{H}$ is spacelike and τ satisfies a convexity property. Why do we expect the τ? We have 1 degree of freedom from embedding because of dimension. τ is then something like an observer. Thus, overall E is more like an energy tensor/vector, and so we define the quasi-local mass to be

$$\inf_{(x,t_0^\mu)} E(\Sigma, X, t_0^\mu)$$

This satisfies the rigidity property. The Euler-Lagrange equation for the quasi-local energy turns out to be a fourth order non-linear elliptic equation of τ. We call this the optimal embedding equation. We use the energy to find the "best matched" space in terms of minimizing the energy for Σ in $\mathbb{R}^{3,1}$.

Properties:

(1) Asymptotics for large and small spheres.
(2) Optimal embedding equation can be solved perturbatively in both cases.
(3) Minimizing properties of critical points near large and small sphere data (Miao-Tam)

Some applications (Chen-W-Yau)

(1) Invariant mass in GR
(2) Quasi-local conserved quantities

Question: (Classical version by Chrusciel 1988): Suppose M_1 and M_2 are AF initial data sets of order $\alpha > 1/2$. If M_1 and M_2 are in the same AF spacetime, do we have $ADM(M_1) = ADM(M_2)$? (In same end, etc. of course)

Chrusciel proved this for stationary spacetimes. Recall that there is a flat slice in Schwarzschild with induced metric that is flat and 2nd fundamental form

$$\sqrt{2m} r^{-3/2} (-\frac{1}{2} dr^2 + r^2 d\Omega^2).$$

This slice is defined by $t = f(r)$ where $f'(r) = \sqrt{2m/r}/(1 - 2m/r)$. Recall the definition of the ADM mass,

$$ADM = \lim_{r \to \infty} \frac{1}{16\pi} \int_{\Sigma_r} (g_{ij,j} - g_{jj,i}) v^i \, d\Sigma_r$$

The mass of this slice is 0 [since the metric is flat]. Does the notion of mass still make sense? However, this slice does not satisfy the AF assumption since it is the borderline case, i.e. $P_{ij} \approx r^{-3/2}$. However, the limit of this quasi-local mass along large surfaces approaches the correct mass.

For initial data sets that violate $\alpha > \frac{1}{2}$, does there exist a formula of total mass depending on (g_{ij}, P_{ij}) that gives the same mass for hypersurfaces in Schwarzschild spacetime?

Question: (Chrusciel 1988): Can you construct a total mass for initial data sets in terms of (g_{ij}, P_{ij}) such that if M_1 and M_2 are in the spacetime, that they have the same mass?

We check this on a dynamical spacetime. The model is the one constructed by Christodoulou and Klainerman in nonlinear stability of Minkowski space. In this case, we again have (t, r, u^1, u^2). Look at hypersurfaces give by $t = f(r, u^1, u^2)$. If $|f| < Cr$ where C is a constant, with $0 < C < 1$, then the limit of the quasi-local mass as we go to ∞ is always the same for these slices. We only make this assumption on f to assure that the slice is a timelike hypersurface.

If $f = t + u$ (i.e. it defines null hypersurfaces), for a constant u, we recover the Bondi mass loss formula.

We can check that for AF initial data sets with $\alpha > 1/2$, the ADM mass is the same as the total mass that is defined by this quasi-local mass.