[redacted] Bieri - Gravitational Radiation

Fig 1

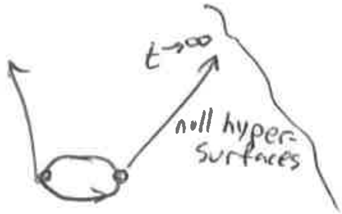


Fig 2

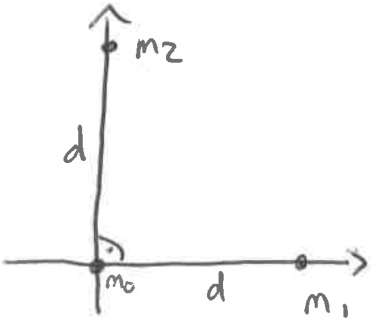
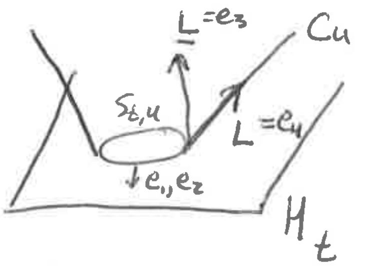


Fig 3:



null frame is $\{e_1, e_2, \underline{L}, L\}$

GRAVITATIONAL RADIATION - A GEOMETRIC-ANALYTIC APPROACH

LYDIA BIERI

"Gravitational Waves" slide: See figure 1.

See figure 2: For a laser interferometer, the masses m_1, m_2 are displaced with respect to m_0 as the waves pass through. These are "instantaneous displacement." There is also a permanent displacement, which is about 2 orders of magnitude smaller, as shown by Christodoulou, when the fully nonlinear aspects of the equations are considered.

"The (t, u) foliations" slide: See figure 3.

Next slide: $\tau_- = \sqrt{1 + u^2}$.

"Important geometric properties" slide: We also have shears $\hat{\chi}$ and $\underline{\hat{\chi}}$.

"Method as introduced by" slide : For a null fluid, $T^{ij} = N^2 Y^i Y^j$.

"Permanent displacement formula" slide: A $\circ$ over a differential operator means the derivative is at the sphere at infinity.

End: We don't know what happens if the masses of the neutrinos are non-zero, which would imply they aren't traveling at the speed of light.

There is a memory effect in just pure Maxwell-Klein-Gordon (in special relativity).

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Gravitational Radiation: A Geometric-Analytic Approach

- Geometries of Physical Spacetimes
- Geometry of Solutions to Einstein Vacuum Equations and Einstein Equations Coupled to other Fields
- Investigating Spacetimes at Null Infinity
- Observing Gravitational Waves
- New Results on Neutrino Radiation

Spacetimes

We consider

Spacetimes (M, g), where M a 4-dimensional manifold with Lorentzian metric g solving Einstein equations:

$$\mathbf{G}_{\mu\nu} := \mathbf{R}_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{g}_{\mu\nu}\,\mathbf{R} = \mathbf{2}\ \mathbf{T}_{\mu\nu}\ , \tag{1}$$

where

$\mathbf{G}_{\mu\nu}$ is the **Einstein tensor**,

$\mathbf{R}_{\mu\nu}$ is the **Ricci curvature tensor**,

$\mathbf{R}$ the **scalar curvature tensor**,

$\mathbf{g}$ the **metric tensor** and

$\mathbf{T}_{\mu\nu}$ denotes the **energy-momentum tensor**.

Definition. *A* **Lorentzian metric** g *is a continuous assignment of a non-degenerate quadratic form* g_p*, of index* 1*, in* T_pM *at each* $p \in M$*.*

For parts of the discussion we concentrate on the Einstein-Vacuum equations.

Solutions of the **Einstein-Vacuum (EV) equations**:

$$R_{\mu\nu} \quad = \quad 0 \; . \tag{2}$$

Spacetimes (M, g), where M is a four-dimensional, oriented, differentiable manifold and g is a Lorentzian metric obeying (2).

Depending on the **matter and energy** present
$\Rightarrow$ **specify** corresponding **equations**.

Construct spacetime in an evolution problem of the Einstein equations.

An **initial data set** consists of

• a 3-dimensional manifold H,
• a complete Riemannian metric $\bar{g}$,
• a symmetric 2-tensor k,
• and a well specified set of initial conditions corresponding to the matter-fields.

These have to satisfy the **constraint equations**.

A **Cauchy development** of an initial data set is
• a globally hyperbolic spacetime (M, g) verifying the Einstein equations
• and an imbedding $i :\rightarrow M$ such that $i_*(\bar{g})$ and $i_*(k)$ are the first and second fundamental forms of $i(H)$ in M.

Consider **asymptotically flat initial data sets**:

• outside a sufficiently large compact set K, $H \backslash K$ is diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$

• and admits a system of coordinates in which $\bar{g} \to \delta_{ij}$ and $k \to 0$ fast enough.

Different Situations

In **[CK] , [Z]**: strongly asymptotically flat initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighbourhood of infinity such that,
as $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \to \infty$:

$$\bar{g}_{ij} = (1 + \frac{2M}{r}) \, \delta_{ij} + o_4 \, (r^{-\frac{3}{2}}) \tag{3}$$

$$k_{ij} = o_3 \, (r^{-\frac{5}{2}}) \, , \tag{4}$$

where M denotes the mass.

In **[B]**: Asymptotically flat initial data $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\bar{g}_{ij} = \delta_{ij} + o_3 \, (r^{-\frac{1}{2}}) \tag{5}$$

$$k_{ij} = o_2 \, (r^{-\frac{3}{2}}) \, . \tag{6}$$

$\Rightarrow$ **Can compute gravitational radiation for the cases [CK], [Z], but not for [B].**
Decay!

Evolution equations of a maximal foliation:

$$
\begin{aligned}
\frac{\partial \bar{g}_{ij}}{\partial t} &= -2\Phi k_{ij} \\
\frac{\partial k_{ij}}{\partial t} &= -\nabla_i \nabla_j \Phi \; + \; (R_{ij} \; - \; 2 k_{im} k^m_{\ j})\Phi
\end{aligned}
$$

Constraint equations of a maximal foliation:

$$
\begin{aligned}
trk &= 0 \\
\nabla^i \, k_{ij} &= 0 \\
R &= \mid k \mid^2
\end{aligned}
$$

Lapse equation of a maximal foliation:

$$
\triangle \Phi \, - \, \mid k \mid^2 \Phi \; = \; 0
$$

Gravitational Waves

What is a gravitational wave?

$\Rightarrow$ **Fluctuation of curvature of the spacetime** propagating as a wave.

Gravitational waves: Localized disturbances in the geometry propagate at the speed of light.

Geometric Analysis $\Leftrightarrow$ Physics

Observation of Gravitational Waves

'**We** -the **observers**- are sitting at **null infinity**.'

$\Rightarrow$ Understand **geometry of spacetimes at null infinity**: Investigate and compute **null asymptotics** of solutions of the Einstein equations, null asymptotic behavior of curvature components and geometric quantities.

$\Rightarrow$ Understand **gravitational radiation**

$\Rightarrow$ Detect **gravitational waves**

• **Nonlinear memory effect (D. Christodoulou, 1991) in regime of Einstein vacuum equations (with large data)**

• **Nonlinear memory effect in regime of Einstein-Maxwell equations (L. Bieri, P. Chen, S.-T. Yau, 2010)**

• **Nonlinear memory effect and neutrino radiation (L. Bieri, D. Garfinkle, recent)**

Christodoulou-Klainerman result
'The global nonlinear stability of the Minkowski space'
([CK], 1993)

$\Rightarrow$ **describes precisely asymptotic behavior at null and timelike infinity.**

This result established that under 'suitable' assumptions on the initial data, i.e. under a smallness assumption, the initial data yield a geodesically complete spacetime.

However, as we want to observe 'from null infinity', we need 'only' investigate the null asymptotics. The results for **null** infinity are **independent from the smallness** assumption.

$\Rightarrow$ Can have **large data**.

The (t, u) **foliations** of the spacetime define a codimension 2 foliation by 2-surfaces

$$S_{t,u} \;=\; H_t \;\cap\; C_u \;, \tag{7}$$

the intersection between H_t (foliation by t) and a u-null-hypersurface C_u (foliation by u).

Null pairs consisting of 2 future-directed null vectors e_4 and e_3 orthogonal to $S_{t,u}$ with e_4 tangent to C_u and

$$\Big\langle e_4, e_3 \Big\rangle \;=\; -\,2 \;. \tag{8}$$

A null pair together with an orthonormal frame e_1, e_2 on $S_{t,u}$ forms a **null frame**.

The **null decomposition** of a tensor relative to a null frame e_4, e_3, e_2, e_1 is obtained by taking **contractions** with the vectorfields e_4, e_3.

Let L and $\underline{L}$ be the outgoing, respectively incoming, null normals to the surface $S_{t,u} = H_t \cap C_u$, for which the component along T is equal to T. Also, the integral curves of L are the null geodesic generators of the null hypersurfaces C_u parametrized by t.

Then T is expressed as

$$T \ = \ \frac{1}{2} \left(L \ + \ \underline{L} \right) \quad . \tag{9}$$

The generator S of scalings is defined to be:

$$S \ = \ \frac{1}{2} \left(\underline{u} \ L \ + \ u \ \underline{L} \right) \quad . \tag{10}$$

And the generator K of inverted time translations is defined as:

$$K \ = \ \frac{1}{2} \left(\underline{u}^2 \ L \ + \ u^2 \ \underline{L} \right) \quad . \tag{11}$$

Then the vectorfield $\bar{K} = K + T$ reads as:

$$\bar{K} \ = \ \frac{1}{2} \left(\tau_+^2 \ L \ + \ \tau_-^2 \ \underline{L} \right) \quad . \tag{12}$$

We denote

$$\begin{aligned} \underline{u} &= u + 2r \\ \tau_- &= \sqrt{1 + u^2} \\ \tau_+ &= \sqrt{1 + \underline{u}^2} \quad . \end{aligned}$$

In [CK], D. Christodoulou and S. Klainerman achieve

- **Proof of existence and uniqueness of solutions, global result**
- **Asymptotic behaviour: Precise description**

Null decomposition of the Riemann curvature tensor of an E-V spacetime:

$$R_{A3B3} = \underline{\alpha}_{AB} \quad (13)$$

$$R_{A334} \quad (14)$$

$$R_{3434} \quad (15)$$

$$^*R_{3434} \quad (16)$$

$$R_{A434} \quad (17)$$

$$R_{A4B4} \quad (18)$$

The null components have the
decay properties:

$$\begin{aligned}
\underline{\alpha} &= O\,(r^{-1}\ \tau_-^{-\frac{5}{2}}) \\
R_{A334} &= O\,(r^{-2}\ \tau_-^{-\frac{3}{2}}) \\
R_{3434} &= O\,(r^{-3}) \\
^*R_{3434} &= O\,(r^{-3}\ \tau_-^{-\frac{1}{2}}) \\
R_{A434}, R_{A4B4} &= o\,(r^{-\frac{7}{2}})
\end{aligned}$$

From the main theorem in [CK], the authors derive the **limiting behavior** of the **curvature components** along the **null** hypersurfaces $\mathbf{C_u}$ as $\mathbf{t} \to \infty$.

$$\lim_{C_u, t \to \infty} r\underline{\alpha} \quad = \quad A(u)$$

with A being a symmetric trace-free 2-covariant tensorfield on S^2.

Correspondingly, the components

$R_{A334}, R_{3434}, \ {}^*R_{3434} \Rightarrow$ tend to **limits** on S^2,

all of which are depending on u.

The components $R_{A434}, R_{A4B4} \Rightarrow$ tend to **zero**.

The limits are shown to have appropriate **decay** as $\mid u \mid \to \infty$:

$$A \quad = \quad o\ (\mid u \mid^{-\frac{5}{2}})$$

The remaining limits have less decay.

Important Geometric Quantities in the Measurement of Gravitational Waves

Fundamental form χ of S relative to C:

$$\chi(X,Y) \ = \ g(D_X L, Y)$$

for any pair of vectors $X, Y \in T_p S$ and L generating vectorfield of C.

Also

$$\underline{\chi}(X,Y) \ = \ g(D_X \underline{L}, Y)$$

Shear $\hat{\chi}$ Traceless part of χ.

Torsion ζ.

$$\zeta \ (\Pi X) \ = \ g \ (Z, \ X)$$

for all X in $T_p M$, where Π denotes the projection to $T_p S$ with $p \in S$ and

$$Z \ = \ - \frac{1}{2} \ D_L \underline{L}$$

where $\underline{L}$ is the generator of the interior cone.

Method as introduced by D. Christodoulou and S. Klainerman in **[CK]**: **Treating propagation equations** along the cones C_u **coupled** to **elliptic systems** on the surfaces $S_{t,u}$.

Further developments in Zipser's proof **[Z]** and in Bieri's proof **[B]**.
In **[Z]** $\Rightarrow$ **electromagnetic field** is present
In **[B]** $\Rightarrow$ **details different** and **borderline cases**
In **Bieri, Garfinkle [BG]** $\Rightarrow$ **neutrino radiation** via **null fluid**

Propagation equation for $tr\chi$:
For **Einstein-Null-Fluid (ENF)**:

$$\frac{dtr\chi}{ds} = -\frac{1}{2}(tr\chi)^2 - |\widehat{\chi}|^2 - 8\pi\mathcal{N}_3^2. \tag{19}$$

For **Einstein-Maxwell (EM)**:

$$\frac{dtr\chi}{ds} + \frac{1}{2}(tr\chi)^2 = -|\widehat{\chi}|^2 - |\alpha(F)|^2$$

The Gauss equation reads

$$K = -\frac{1}{4}tr\chi tr\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}} - \text{“}W\text{”} + \text{ contributions from } T$$

Here “W” denotes a component of the Weyl curvature other than $\underline{\alpha}$.

Define the function $\underline{\mu}$ as

$$\underline{\mu} = -\mathrm{d}\!\!/\mathrm{iv}\,\underline{\zeta} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}} - \text{"}W\text{"} + \text{ contributions from } T$$

The latter, with the help of the Gauss curvature K, can be written as

$$\underline{\mu} = -\mathrm{d}\!\!/\mathrm{iv}\,\underline{\zeta} + K + \frac{1}{4}tr\chi tr\underline{\chi}. \tag{20}$$

The null Codazzi and conjugate null Codazzi equations read

$$\begin{aligned} \mathrm{d}\!\!/\mathrm{iv}\,\widehat{\chi} &= -\widehat{\chi}\cdot\zeta + \frac{1}{2}(\nabla\!\!\!/\, tr\chi + \zeta tr\chi) - \text{"}W\text{"} \\ &\quad + \text{contributions from } T \end{aligned}$$

$$\begin{aligned} \mathrm{d}\!\!/\mathrm{iv}\,\underline{\widehat{\chi}} &= \underline{\widehat{\chi}}\cdot\zeta + \frac{1}{2}(\nabla\!\!\!/\, tr\underline{\chi} - \zeta tr\underline{\chi}) + \text{"}W\text{"} \\ &\quad + \text{contributions from } T \end{aligned}$$

Null Asymptotics $\Rightarrow$ Gravitational Radiation

Limit for the shear $\hat{\chi}$

$$\lim_{C_u, t \to \infty} r^2 \widehat{\chi} \;=\; \Sigma(u)$$

Σ symmetric trace-free 2-covariant tensorfield on S^2 depending on u.

Moreover,

$$\begin{aligned} \lim_{C_u, t \to \infty} r \; tr\chi \quad &= \quad - \lim_{C_u, t \to \infty} r \; tr\underline{\chi} \;=\; 2 \\ \lim_{C_u, t \to \infty} r \widehat{\underline{\chi}} \quad &= \quad \Xi(u) \end{aligned}$$

Ξ symmetric trace-free 2-covariant tensorfield on S^2 depending on u.

$$\Xi \;=\; o\,(|\, u \,|^{-\frac{3}{2}}) \quad \text{as} \quad |\, u \,| \to \; \infty \quad .$$

Investigate Null Geometry

Null hypersurfaces => asymptotics as $t \to \infty$

Geometry and Analysis $\Rightarrow$ Physics of Radiation

Quantities that eventually play the leading roles:

shears, torsion,
particular curvature component(s,)
particular component(s) of energy-momentum tensor.

Ψ component of a second fundamental form or torsion,
Φ component of the Weyl curvature,
T component of the energy-momentum tensor.

$$\nabla\!\!\!/_N \hat{\chi} = \text{``}\Psi\Psi\text{''} + \text{``}\Psi\hat{\otimes}\Psi\text{''} + \text{``}\nabla\!\!\!/\,\hat{\otimes}\Psi\text{''} + \text{``}\Phi\text{''} + \text{``}T\text{''}$$

$$\nabla\!\!\!/_N \hat{\eta} + \frac{1}{2} tr\theta\hat{\eta} = \text{``}\Psi\Psi\text{''} + \text{``}\Psi\hat{\otimes}\Psi\text{''} + \text{``}\nabla\!\!\!/\,\hat{\otimes}\Psi\text{''} + \text{``}\Phi\text{''} + \text{``}T\text{''}$$

In the situation when an **electromagnetic field** is present:

$\Rightarrow$ Multiply the first equation by r^2 and the second by r:

$$\not\nabla_N(r^2\widehat{\chi}) = -r\widehat{\eta} + O(r^{-1})$$

$$\not\nabla_N(r\widehat{\eta}) = -r\underline{\alpha} + O(r^{-1})$$

Take the limits on C_u as $t \to \infty$:

$$\frac{\partial}{\partial u}\Sigma = -\Xi$$

$$\frac{\partial}{\partial u}\Xi = -\frac{1}{4}A_W$$

Hawking mass $m(t,u)$ contained in a surface $S_{t,u}$ defined as:

$$m(t,u) \;=\; \frac{r}{2}\left(1 \;+\; \frac{1}{16\pi}\;\int_{S_{t,u}} tr\chi \; tr\underline{\chi}\right) \tag{21}$$

Bondi mass $M(u)$ contained in C_u defined as:

$$M(u) \;=\; \lim_{t\to\infty} m(t,u) \tag{22}$$

Investigating

$$\frac{\partial}{\partial t} m(t,u)$$

and

$$\frac{\partial}{\partial u} m(t,u)$$

and taking corresponding limits.

For physically interesting spacetimes

$$m(t,u) = M(u) + O(r^{-1})$$

Bondi mass loss formulas

[CK] derived and used in **[C]**:

$$\frac{\partial M}{\partial u} \;=\; \frac{1}{8\pi} \int_{S^2} |\ \Xi\ |^2 \;\; d\mu_{\gamma_{S^2}}$$

[Z] derived used in **[BCY]**:

$$\frac{\partial}{\partial u} M\,(u) = \frac{1}{8\pi} \int_{S^2} \left(|\Xi|^2 + \frac{1}{2} \,|A_F|^2 \right) d\mu_{\overset{\circ}{\gamma}}$$

[BG] derived and used:

$$\frac{\partial}{\partial u} M\,(u) = \frac{1}{8\pi} \int_{S^2} \left(|\Xi|^2 + 4\pi T^*_{\underline{LL}} \right) d\mu_{\overset{\circ}{\gamma}}$$

[C] introduced

$$F \;=\; \frac{1}{8} \int_{-\infty}^{\infty} \mid \Xi(u) \mid^2 \; du \qquad (23)$$

with $\mathbf{F}/4\pi$ the **total energy radiated to infinity in a given direction, per unit solid angle**.

⇒ Derived **nonlinear memory effect of gravitational waves [Christodoulou]**

[BCY] introduced

$$F = \frac{1}{8} \int_{-\infty}^{+\infty} \left(| \, \Xi \, |^2 + \frac{1}{2} \, | \, A_F \, |^2 \right) du \quad . \tag{24}$$

⇒ Derived **nonlinear electromagnetic Christodoulou memory effect**
[Bieri-Chen-Yau]

[BG] introduced

$$F = \frac{1}{8} \int_{-\infty}^{+\infty} \left(| \, \Xi \, |^2 + 4\pi T^*_{\underline{LL}} \right) du \quad . \tag{25}$$

⇒ Derived **nonlinear Christodoulou memory effect for neutrino radiation via null fluid**
[Bieri-Garfinkle]

With this energy consider equation

$$\not{div}\ (\Sigma^{+} - \Sigma^{-}) = \not{\nabla} f$$

where f is a solution of

$$\not{\triangle} f = 2\,(F - \bar{F})\ , \quad \overline{f} = 0$$

$\not{\nabla}, \not{div}, \not{\triangle}$ on S^2. Integrability condition of the last two equations is that F is L^2-orthogonal to the first eigenspace of $\not{\triangle}$:

$$F_{(1)} = 0\ .$$

Derive

$$\Sigma^{+} \; - \; \Sigma^{-} \; = \; \frac{1}{2} \int_{-\infty}^{\infty} \Xi(u) \; du \tag{26}$$

and

$$\Sigma(u) \; = \; \Sigma^{-} \; + \; \frac{1}{2} \int_{-\infty}^{u} \Xi(u') \; du'$$

$$\mathbf{\Sigma(u) - \Sigma^{-}}$$

related to

instantaneous displacements of faraway test masses w.r.t. reference test mass, relative to which they are initially at rest.

$$\mathbf{\Sigma^{+} - \Sigma^{-}}$$

yields

permanent displacement of the test masses.
Non-linear effect.
An effect observable in principle.

Investigate

$$\Sigma(u) - \Sigma^{-} \quad \text{and} \quad \Sigma^{+} - \Sigma^{-}$$

for physically relevant spacetimes.

1. Keep general T_{ij}.

2. Investigate EV, EM, ENF.

3. Geometry of resulting spacetimes

$\Rightarrow$ Gravitational radiation in astrophysical scenarios.

Permanent Displacement Formula

Christodoulou's Memory Effect $\Rightarrow$ governed by the permanent displacement formula $\Sigma^{+} - \Sigma^{-}$.

Theorem 1. *Let* $\Sigma^{+}(\cdot) = \lim_{u \to \infty} \Sigma(u, \cdot)$ *and* $\Sigma^{-}(\cdot) = \lim_{u \to -\infty} \Sigma(u, \cdot)$ *In the following, let* $|S|^2$ *denote a component of the energy-momentum tensor with "right" decay. (*S *can be a tensor or a function, depending on the fields.) Moreover, let "*T*" denote lower order components of the energy-momentum tensor, where these can be quadratics of the fields. Let*

$$F(\cdot) = \int_{-\infty}^{\infty} \big(\mid \Xi(u, \cdot) \mid^2 + c_1 \mid S(u, \cdot) \mid^2 \big) du \quad . \tag{27}$$

Moreover, let Φ *be the solution with* $\bar{\Phi} = 0$ *on* S^2 *of the equation*

$$\overset{\circ}{\triangle\!\!\!/} \ \Phi = F - \bar{F} \quad .$$

Then $\Sigma^{+} - \Sigma^{-}$ *is given by the following equation on* S^2*:*

$$\overset{\circ}{di\!\!/v} \ (\Sigma^{+} - \Sigma^{-}) = \overset{\circ}{\nabla\!\!\!/} \ \Phi \quad . \tag{28}$$

Proof - Sketch: We have

$$\Sigma(u) = \Sigma^{-} - \int_{-\infty}^{u} \Xi(u')du'$$

and

$$\Sigma^{+} - \Sigma^{-} = - \int_{-\infty}^{\infty} \Xi(u')du' \quad .$$

Consider the normalized null Codazzi equation

$$(\not{d}iv\,\widehat{\chi})_A - \frac{1}{2}\not{\nabla}_A tr\chi + \epsilon_B\widehat{\chi}_{AB} - \frac{1}{2}\epsilon_A tr\chi \quad = \quad -\beta(W)_A + \text{``}t\text{''} \qquad (29)$$

Multiply equation (29) by r^3 and take the limit as $t \to \infty$ on C_u.

Let

$$E = \lim_{C_u, t\to\infty} \left(r^2\epsilon\right)$$

We **obtain**

$$\overset{\circ}{\not{d}iv}\ \mathbf{\Sigma} = \overset{\circ}{\not{\nabla}}\ \mathbf{H} + \mathbf{E}$$

H denotes

$$H = \lim_{C_u, t\to\infty} \left(r^2(tr\chi' - \frac{2}{r})\right) \ .$$

It can be shown that

$$\frac{\partial H}{\partial u} = 0 \ .$$

$\Rightarrow$ **Focus** on E.

Hodge system for ϵ

$$
\begin{aligned}
d\!\!\!/iv\,\epsilon &= -\nabla_N \delta - \frac{3}{2} tr\theta\delta + \hat{\eta}\cdot\hat{\theta} \\
&\quad -2(a^{-1}\nabla\!\!\!/\, a)\cdot\epsilon + c\,\text{“}t\text{”} - c \mid s \mid^2 \\
c\!\!\!/url\ \epsilon &= \sigma(W) + \hat{\theta}\wedge\hat{\eta} \quad .
\end{aligned}
$$

We work with the following

$$
\begin{aligned}
\triangle\Psi &= r \mid \hat{\eta} \mid^2 - c \mid s \mid^2 \\
\triangle\Psi' &= -ra^{-1}\lambda\Big(\mid \hat{\eta} \mid^2 - \overline{\mid \hat{\eta} \mid^2}\Big) \\
&\quad + \frac{r^2 a^{-1}}{4}\Big(a D\!\!\!/_4 \mid s \mid^2 - \overline{a D\!\!\!/_4 \mid s \mid^2}\Big)
\end{aligned}
$$

whereas in the EV case treated in [CK] it is

$$
\begin{aligned}
\triangle\Psi &= r \mid \hat{\eta} \mid^2 \\
\triangle\Psi' &= -ra^{-1}\lambda\Big(\mid \hat{\eta} \mid^2 - \overline{\mid \hat{\eta} \mid^2}\Big) \quad .
\end{aligned}
$$

We derive

$$
\begin{aligned}
d\!\!\!/iv\,\epsilon &= \rho - \bar{\rho} + \hat{\chi}\cdot\hat{\eta} - \overline{\hat{\chi}\cdot\hat{\eta}} + r^{-1}\triangle\!\!\!/\,\Psi - r^{-2}\nabla_N\Psi' \\
&\quad -r^{-3}a^{-1}\lambda\Psi' + l.o.t. \qquad (30) \\
c\!\!\!/url\ \epsilon &= \sigma(W) + \hat{\theta}\wedge\hat{\eta} \qquad (31)
\end{aligned}
$$

Continue from

$$
\begin{aligned}
\not{d}\!iv\,\epsilon &= \rho - \bar{\rho} + \widehat{\chi}\cdot\widehat{\eta} - \overline{\widehat{\chi}\cdot\widehat{\eta}} + r^{-1}\not{\Delta}\,\Psi - r^{-2}\nabla_N\Psi' \\
&\quad - r^{-3}a^{-1}\lambda\Psi' + l.o.t. \qquad (30) \\
\not{c}url\ \epsilon &= \sigma(W) + \widehat{\theta}\wedge\widehat{\eta} \qquad (31)
\end{aligned}
$$

We compute the following limits in the new situation.

$$
\lim_{C_u, t\to\infty} \Psi = \boldsymbol{\Psi} \qquad \lim_{C_u, t\to\infty} \Psi' = \boldsymbol{\Psi}'
$$

$$
\lim_{C_u, t\to\infty} r\nabla_N\Psi = \Omega(u,\cdot) \qquad \lim_{C_u, t\to\infty} r\nabla_N\Psi' = \Omega'(u,\cdot).
$$

Multiply equations (30) and (31) by r^3 and take the limits on C_u as $t \to \infty$. This yields:

$$
\begin{aligned}
\overset{\circ}{\not{c}url}\ E &= Q + \Sigma\wedge\Xi \qquad (32) \\
\overset{\circ}{\not{d}iv}\ E &= P - \bar{P} + \Sigma\cdot\Xi - \overline{\Sigma\cdot\Xi} \\
&\quad + \overset{\circ}{\not{\Delta}}\ \boldsymbol{\Psi} - \boldsymbol{\Psi}' - \Omega'\ . \qquad (33)
\end{aligned}
$$

Investigate limits as $u \to +\infty$ and $u \to -\infty$.

$\Rightarrow$

E tends to a limit E^+ as $u \to +\infty$ and to E^- as $u \to -\infty$.

Have

$$
\begin{aligned}
\overset{\circ}{\not{c}url}\ \ E &= Q + \Sigma \wedge \Xi &(32)\\
\overset{\circ}{\not{d}iv}\ \ E &= P - \bar{P} + \Sigma \cdot \Xi - \overline{\Sigma \cdot \Xi} \\
&\quad + \overset{\circ}{\not{\triangle}}\ \Psi - \Psi' - \Omega' \quad . &(33)
\end{aligned}
$$

Obtain

$$
\overset{\circ}{\not{c}url}\ \ (E^{+} - E^{-}) \ = \ 0
$$

Now, compute $\overset{\circ}{\not{d}iv}\ (E^{+} - E^{-})$.

Have to consider the corresponding limits for the terms involving $\mathbf{\Psi}$ and $\mathbf{\Psi}'$, that is also Ω'.

We use the fact that

$$
\not{D}_4 \mid s \mid^2 = -tr\chi \mid s \mid^2 + l.o.t. \tag{34}
$$

Using (34), (30), we deduce formulas for $\mathbf{\Psi}$, $\mathbf{\Psi}'$, Ω, Ω' by computing the limits in (33).

Evaluating the difference of the limits as $u \to +\infty$ and $u \to -\infty$ in (33), the contribution of $\overset{\circ}{\not\Delta}\ \Psi$, Ψ' and Ω' comes only from terms in Ω'. We find that Ω' tends to limits $\Omega'^{+}(\cdot)$ and $\Omega'^{-}(\cdot)$ as $t \to \infty$ and $t \to -\infty$, respectively. Thus, we conclude

$$
\begin{aligned}
\Omega'^{+}(\cdot) - \Omega'^{-}(\cdot) \quad = \quad & \int_{-\infty}^{+\infty} \Big(\ \mid \Xi(u,\cdot) \mid^2 - \overline{\mid \Xi(u,\cdot) \mid^2} \\
& + c_1 \mid S(u,\cdot) \mid^2 - c_1 \overline{\mid S(u,\cdot) \mid^2} \ \Big) du \quad .
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
\overset{\circ}{d\not{i}v}\ (E^{+} - E^{-}) \quad = \quad & -\Omega'^{+} + \Omega'^{-} \\
= \quad & \int_{-\infty}^{+\infty} \Big(- \mid \Xi(u,\cdot) \mid^2 + \overline{\mid \Xi(u,\cdot) \mid^2} \\
& - c_1 \mid S(u,\cdot) \mid^2 + c_1 \overline{\mid S(u,\cdot) \mid^2} \ \Big) du \quad .
\end{aligned} \tag{35}
$$

We know that

$$(E^{+} - E^{-}) = \overset{\circ}{\nabla\!\!\!/} \ \Phi \tag{36}$$

with Φ being the solution of vanishing mean of

$$\overset{\circ}{\triangle\!\!\!/} \ \Phi = -\Omega'^{+} + \Omega'^{-} \quad \text{on } S^2 \quad .$$

Collect the results :

- $$\overset{\circ}{di\!\!\!/v} \ \Sigma = \overset{\circ}{\nabla\!\!\!/} \ H + E$$

with $\frac{\partial H}{\partial u} = 0$.

- $$\overset{\circ}{cu\!\!\!/rl} \ (E^{+} - E^{-}) = 0$$

- $$\begin{aligned} \overset{\circ}{di\!\!\!/v} \ (E^{+} - E^{-}) &= -\Omega'^{+} + \Omega'^{-} \\ &= \int_{-\infty}^{+\infty} \Big(- \mid \Xi(u, \cdot) \mid^2 + \overline{\mid \Xi(u, \cdot) \mid^2} \\ &\quad -c_1 \mid S(u, \cdot) \mid^2 + c_1 \overline{\mid S(u, \cdot) \mid^2} \ \Big) du \quad . \end{aligned}$$

We conclude

$$\overset{\circ}{di\!\!\!/v} \ (\Sigma^{+} - \Sigma^{-}) = E^{+} - E^{-} \quad . \tag{37}$$

This proves theorem 1.

Einstein-Maxwell Case

What happens in the presence of an electromagnetic field?

Einstein-Maxwell equations:

$$\mathbf{G}_{\mu\nu} := \mathbf{R}_{\mu\nu} - \frac{1}{2}\,\mathbf{g}_{\mu\nu}\,\mathbf{R} = 8\pi\ \mathbf{T}_{\mu\nu}\ , \tag{38}$$

setting $G = c = 1$, $\mu, \nu = 0, 1, 2, 3$, where
$\mathbf{G}_{\mu\nu}$ is the **Einstein tensor**,
$\mathbf{R}_{\mu\nu}$ is the **Ricci curvature tensor**,
$\mathbf{R}$ the **scalar curvature tensor**,
$\mathbf{g}$ the **metric tensor** and
$\mathbf{T}_{\mu\nu}$ denotes the **stress-energy tensor of the electromagnetic field**.

In particular, $\mathbf{F}$ denoting the **electromagnetic field**, the tensor $\mathbf{T}_{\alpha\beta}$ reads:

$$T_{\alpha\beta} = \frac{1}{8\pi}\Big(F_{\alpha}{}^{\rho}F_{\beta\rho} - \frac{1}{4}g_{\alpha\beta}F_{\rho\sigma}F^{\rho\sigma}\Big) \tag{39}$$

F is an antisymmetric covariant 2-tensor.

$T_{\mu\nu}$ is trace-free. $\Rightarrow$ Einstein equations (38) become

$$R_{\mu\nu} = 8\pi T_{\mu\nu} \quad . \tag{40}$$

The **Einstein-Maxwell (EM) equations** are given by

$$\begin{aligned} R_{\mu\nu} &= 8\pi T_{\mu\nu} && (41)\\ D^\alpha F_{\alpha\beta} &= 0 && (42)\\ D^\alpha\ {}^*F_{\alpha\beta} &= 0. && (43)\end{aligned}$$

Whereas in the EV case, the **Weyl tensor** satisfies the homogeneous equations

$$D^\alpha W_{\alpha\beta\gamma\delta} = 0 \ ,$$

in the EM case the corresponding equations are inhomogeneous

$$D^\alpha W_{\alpha\beta\gamma\delta} = \frac{1}{2}(D_\gamma R_{\beta\delta} - D_\delta R_{\beta\gamma}) \quad . \qquad (44)$$

Zipser [Z] works with the same conditions as **[CK]** on the metric, second fundamental form and curvature,
in addition she imposes a decay condition on the electromagnetic field, namely

$$F|_H = o_3\left(r^{-\frac{5}{2}}\right). \tag{45}$$

The null components of the electromagnetic field are

$$\begin{array}{ll} F_{A3} = \underline{\alpha}(F)_A & F_{A4} \\ F_{34} & F_{12}. \end{array} \tag{46}$$

The estimates in **[Z]** yield the decay behavior:

$$\begin{aligned} \underline{\alpha}(F) &= O\ (r^{-1}\ |u|^{-\frac{3}{2}}) \\ F_{12}, F_{34} &= O\ (r^{-2}\ |u|^{-\frac{1}{2}}) \\ F_{A4} &= o\ (r^{-\frac{5}{2}}) \end{aligned}$$

$\underline{\alpha}(F)$, F_{12} , F_{34} have limits at null infinity.

Guiding term

$$\Rightarrow \underline{\alpha}(\mathbf{F})$$

Limit

$$\lim_{C_u, t \to \infty} r\underline{\alpha}(F) = A_F(u, \cdot)$$

A_F is a 1-form on S^2 depending on u with decay property:

$$|A_F(u, \cdot)| \;\leq\; (1 + |u|)^{-3/2}$$

Pointwise norms $| \quad |$ of the tensors on S^2 relate to metric $\overset{\circ}{\gamma}$, being the limit of the induced metrics on $S_{t,u}$ $\forall u$ as $t \to \infty$.

Einstein-Null-Fluid and Neutrino Radiation

Describe **burst of neutrinos** as a **null fluid**:

Energy-momentum tensor has the form

$$T_{ij} = \mathcal{N}^2 k_i k_j \tag{47}$$

k a null vector
$\mathcal{N}$ a positive scalar function.

Notation: in what follows:
Denote component $T(X, Y) = T_{ij} X^i Y^j$ of the energy-momentum tensor by T_{XY} for any vectors X, Y on M.

The twice contracted Bianchi identities imply that

$$D^j G_{ij} = 0 \quad . \tag{48}$$

Thus

$$D^j T_{ij} = 0. \tag{49}$$

The **Einstein equations** (38) for a **null fluid** reduce to:

$$R_{ij} = 8\pi T_{ij} \tag{50}$$

Typical **sources** of such neutrino bursts:

core-collapse supernovae

and

binary neutron star mergers.

Initially "at" burst: Neutrinos fly in all directions.

Later: Neutrinos follow the **outgoing null geodesics** generated by $\mathbf{L}$.

The vector k will be of the form

$$k = aL + b\underline{L} + V$$

with V denoting a vector tangent to S.

Can be shown that :

along C_u as $t \to \infty$, $\underline{L}$ and V decay and

$\mathbf{k}$ **finally becomes** $\mathbf{L}$.

Let k be a null geodesic, that is,

$$k^a \nabla_a k = \nabla_k k = 0$$
$$k^a k_a = 0 \ .$$

Excursion to Minkowski spacetime

In Minkowski space there exist conformal Killing fields X, that is,

$$\nabla_{(a} X_{b)} = \phi g_{ab}$$

for some scalar ϕ, that is

$$(\mathcal{L}_X g) = \phi g \ .$$

Then it follows that

$$k^a \nabla_a (X_a k^a) = 0$$

and consequently that for each geodesic there exists a constant c such that $k^a X_a = c$.

Back to Lorentzian spacetime

A Lorentzian manifold in **general** does **not** admit conformal Killing fields.
$\Rightarrow$ Above equations do **not** hold.

However,

Asymptotic flatness $\Rightarrow$ **guarantees**
the existence of **almost- and quasi-conformal Killing fields**.

This means

Deformation tensors are suitably small and tend to zero as $t \to \infty$ in a suitable way.

Then

the afore-mentioned equations 'hold in an asymptotic sense'.

Recall the vectorfields

$$T \;=\; \frac{1}{2}\left(L \;+\; \underline{L}\right) \quad ,$$

$$K \;=\; \frac{1}{2}\left(\underline{u}^2 \; L \;+\; u^2 \; \underline{L}\right) \quad .$$

Deformation tensor for T

$$^{(T)}\pi_{\alpha\beta} = (\mathcal{L}_T g)_{\alpha\beta}. \tag{51}$$

Deformation tensor for K

$$^{(K)}\pi_{\alpha\beta} = (\mathcal{L}_K g)_{\alpha\beta}. \tag{52}$$

For a S-tangential vectorfield V it is

$$\begin{aligned}(\mathcal{L}_K g)(V,V) &= \frac{1}{2}\underline{u}^2(\mathcal{L}_L g)(V,V) + \frac{1}{2}u^2(\mathcal{L}_{\underline{L}} g)(V,V) \\ &= V^A V^B \underline{u}^2 \chi_{AB} + V^A V^B u^2 \underline{\chi}_{AB} \ .\end{aligned}$$

In **[BG]** we derive

$$T_{\underline{L}\underline{L}} = O(r^{-2}\tau_-^{-3})$$

Other components of T are of **lower order**.

Gravitational Wave Experiments

How do these results relate to experiment?

In his derivation of the nonlinear memory effect in the EV case, Christodoulou shows how the **theoretical result on** $\Sigma^+ - \Sigma^-$ leads to an **effect measurable** by a **laser interferometer gravitational-wave detector**.

This effect shows as a **permanent displacement** of the test masses of the detector after a wave train has passed.

Here discuss:

Permanent displacement of the test masses in the neutrino (null fluid) case:
Null fluid comes into the formula $\Sigma^+ - \Sigma^-$**.**

Instantaneous displacement of the test masses in the ENF case: **unchanged**.

3 test masses $\mathbf{m_0}$, $\mathbf{m_1}$, $\mathbf{m_2}$ suspended by equal length pendulums.

$\mathbf{m_0}$: reference mass.

Measure by laser interferometry the distance of $\mathbf{m_1}$ and $\mathbf{m_2}$ from the reference mass $\mathbf{m_0}$

The beam splitter is at m_0.

Motion of masses on the horizontal plane: considered free for timelike scales much shorter than the period of the pendulums.

Any **difference** in the **light travel times** between m_0 and m_1 and m_2, respectively, results in a **difference of phase of the laser light** at m_0.

m_0, m_1, m_2 move along **geodesics** γ_0, γ_1, γ_2 in spacetime.
T: unit future-directed tangent vectorfield of γ_0
t: arc length along γ_0.
Let H_t for each t be the spacelike, geodesic hyperplane through $\gamma_0(t)$ orthogonal to T.

Consider the **orthonormal frame field** (T, E_1, E_2, E_3) along γ_0, where (E_1, E_2, E_3) is an orthonormal frame for H_0 at $\gamma_0(0)$, parallely propagated along γ_0.

$\Rightarrow$ at each t, (E_1, E_2, E_3) is an orthonormal frame for H_t at $\gamma_0(t)$.

Assign to a point p in spacetime, lying in a neighbourhood of γ_0, the cylindrical normal coordinates (t, x^1, x^2, x^3), based on γ_0, if $p \in H_t$ and $p = expX$ with $X = \sum_i x^i E_i \in T_{\gamma_0(t)} H_t$.

In these coordinates we have

$$g_{\mu\nu} \; - \; \eta_{\mu\nu} \; = \; O \; (R \; d^2) \; , \qquad (53)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and:

$$d \; = \mid X \mid = \; \sqrt{\sum_i \; (x^i)^2} \qquad (54)$$

is the distance of p from the center $\gamma_0(t)$ on H_t.

Let τ be the time scale in which the curvature varies significantly.

Then, the **displacements of the masses from their initial positions** will be

$$O(R\tau^2) \quad .$$

Assume that

$$\frac{d}{\tau} << 1 \; . \tag{55}$$

The speed of light can be taken to be 1.

Thus, differences in phase of the laser light will, under this assumption, accurately reflect differences in distance of m_1 and m_2 from m_0.

The same assumption (55) allows us to replace the **geodesic equation** for γ_1 and γ_2 by the **Jacobi equation** (geodesic deviation from γ_0).

$$\frac{d^2 x^k}{dt^2} = - R_{kTlT} \; x^l \tag{56}$$

with

$$R_{kTlT} = R \; (E_k, T, E_l, T) \; . \tag{57}$$

Now, assume for simplicity that the source is in the E_3-direction.

Investigate the formula (56) for the Einstein-Null-Fluid (ENF) situation:

Non-charged test masses: formula (56) stays the same, but the null fluid comes in.
However, it enters at lower order.

It is:

$$R_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma}) - \frac{1}{6}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right)R \quad . \tag{58}$$

$\Rightarrow$

$$R_{k0l0} = W_{k0l0} + \frac{1}{2}(g_{kl}R_{00} + g_{00}R_{kl} - g_{0l}R_{k0} - g_{k0}R_{0l}) \tag{59}$$

The ENF equations tell us:

$$R_{00} = 8\pi T_{00} \quad ,$$

and in particular, we have

$$R_{00} = 8\pi T_{00} = 2\pi(T_{\underline{L}\underline{L}} - T_{LL}) \tag{60}$$

we can investigate the components of the Ricci curvature on the right hand side of (59).
R_{00} includes the term $T_{\underline{L}\underline{L}}$. **Worst decay behavior.**

Consider $L = T - E_3$, $\underline{L} = T + E_3$.

The **leading components of the curvature** are

$$\underline{\alpha}_{AB}(W) \quad = \quad R\ (E_A,\ \underline{L},\ E_B,\ \underline{L}) \tag{61}$$
$$\underline{\alpha}_{AB}(W) \quad = \quad \frac{A_{AB}(W)}{r}\ +\ o\ (r^{-2})\ . \tag{62}$$

The **leading components of the null fluid** are

$$T_{\underline{L}\underline{L}} \quad = \quad \frac{T^*_{\underline{L}\underline{L}}}{r^2}\ +\ o\ (r^{-2})\ . \tag{63}$$

Denote the kth Cartesian coordinate of the mass m_A for $A = 1, 2$ by $x^k{}_{(A)}$.

Then the **Jacobi equation** becomes

$$\frac{d^2 \ x^k{}_{(A)}}{d \ t^2} \ = \ - \ \frac{1}{4} \ r^{-1} \ A_{AB} \ x^l{}_{(B)} \ + \ O \ (r^{-2})$$

that is

$$\begin{aligned} \frac{d^2 \ x^3{}_{(C)}}{d \ t^2} \quad &= \quad 0 \\ \frac{d^2 \ x^A{}_{(C)}}{d \ t^2} \quad &= \quad - \ \frac{1}{4} \ r^{-1} \ A_{AB} \ x^B{}_{(D)} \ + \ O(r^{-2}) \end{aligned}$$

From the Jacobi equation $\Rightarrow$ see that the null fluid enters on the right hand side at order (r^{-2}) only.

$\Rightarrow$ **The null fluid does not contribute at leading order to the deviation measured by the Jacobi equation.**

$\Rightarrow$ At leading order, results for the Einstein vacuum case apply.

Obtain: In the vertical direction there is no acceleration to leading order (r^{-1}).

Initially m_1 and m_2 are at rest at equal distance d_0 and at right angles from m_0. This implies the following initial conditions, as $t \to -\infty$:

$$x^3{}_{(A)} = 0 \ , \ \dot{x}^3{}_{(A)} = 0 \ , \ x^B{}_{(A)} = d_0 \delta^B_A \ , \ \dot{x}^B{}_{(A)} = 0.$$

The right hand side being very small, one can substitute the initial values on the right hand side. Then the motion is confined to the horizontal plane. One has to leading order:

$$\ddot{x}^A{}_{(B)} = -\frac{1}{4} r^{-1} d_0 A_{AB} \quad . \tag{64}$$

One obtains

$$\dot{x}^A{}_{(B)} (t) = -\frac{1}{4} d_0 r^{-1} \int_{-\infty}^{t} A_{AB} (u) \, du \ . \tag{65}$$

In view of equation
$\frac{\partial \Xi}{\partial u} = -\frac{1}{4}\ A$ and $\lim_{|u| \to \infty} \Xi = 0$

we obtain

$$- \int_{-\infty}^{t} A_{AB}\ (u)\ du\ =\ \Xi\ (t) \tag{66}$$

and

$$\dot{x}^{A}{}_{(B)}\ (t)\ =\ \frac{d_0}{r}\ \Xi_{AB}\ (t)\ . \tag{67}$$

As $\Xi \to 0$ for $u \to \infty$, the test masses return to rest after the passage of the gravitational wave.

Taking into account
$\frac{\partial \Sigma}{\partial u} = -\Xi$,

and integrating again:

$$x^{A}{}_{(B)}\ (t)\ =\ -\ (\frac{d_0}{r})\ (\Sigma_{AB}\ (t)\ -\ \Sigma^{-})\ . \tag{68}$$

The limit $t \to \infty$ is taken and it follows that the test masses experience **permanent displacements**.

Thus
$\Sigma^+ - \Sigma^-$
is equivalent to an overall displacement of the test masses:

$$\triangle \; x^A{}_{(B)} \;=\; - \; (\frac{d_0}{r}) \; (\Sigma^+_{AB} \; - \; \Sigma^-_{AB}) \; . \qquad (69)$$

The right hand side of (69) includes terms from the **null fluid** at highest order as given in our theorem 1.

Recall also: total energy $\frac{\mathbf{F}}{4\pi}$ radiated to infinity in a given direction per unit solid angle:

$$F = \int_{-\infty}^{+\infty} \Big(\mid \Xi \mid^2 + 4\pi T^*_{\underline{LL}} \Big) \, du \quad .$$

Derive

$$\Sigma^{+} \; - \; \Sigma^{-} \; = \; \frac{1}{2} \int_{-\infty}^{\infty} \Xi(u) \; du \qquad (70)$$

and

$$\Sigma(u) \; = \; \Sigma^{-} \; + \; \frac{1}{2} \int_{-\infty}^{u} \Xi(u') \; du'$$

$$\mathbf{\Sigma(u) - \Sigma^{-}}$$

related to

instantaneous displacements of faraway test masses w.r.t. reference test mass, relative to which they are initially at rest.

$$\mathbf{\Sigma^{+} - \Sigma^{-}}$$

yields

permanent displacement of the test masses.
Non-linear effect.
An effect observable in principle.

Now: Denote the direction of observation by $\xi \in S^2 \subset \mathbb{R}^3$,

Let X, Y be arbitrary vectors lying in the tangent plane at ξ, i.e. in $T_\xi S^2$.

Let Π be the projection to the plane through the origin orthogonal to ξ.

$< \, , \, >$ denotes the inner product.

The **solution at the observation point** ξ is expressed as an integral over S^2 of a contribution from each $\xi' \in S^2$:

$$(\Sigma^+ \; - \; \Sigma^-) \; (X, \; Y) =$$

$$-\frac{1}{2\pi} \int_{\xi' \in S^2} (F - F_{[1]})(\xi') \frac{< X, \xi' >< Y, \xi' > - \frac{1}{2} < X, Y >| \; \Pi\xi' \; |^2}{1- < \xi, \xi' >} d\mu_{\mathring{\gamma}}(\xi')$$

Subscript [1] denotes the projection onto the sum of the 0^{th} ($l = 0$) and 1^{st} ($l = 1$) eigenspaces of $\overset{\circ}{\Delta}$. Multiplicity of the lth eigenspace $2l + 1$, eigenvalue $l(l + 1)$.

Recall

$$F \; = \; \frac{1}{8} \int_{-\infty}^{+\infty} \; | \; \Xi(u) \; |^2 \;\; du \quad \text{(EV)}$$

$$F = \frac{1}{8} \int_{-\infty}^{+\infty} \left(| \; \Xi \; |^2 + \frac{1}{2} \; | \; A_F \; |^2 \right) du \quad \text{(EM)}$$

$$F = \frac{1}{8} \int_{-\infty}^{+\infty} \left(| \; \Xi \; |^2 + 4\pi \mathcal{N}_1^{*2} \right) du \quad \text{(ENF)}$$

and

$$\Sigma^+ \; - \; \Sigma^- \; = \; - \; \frac{1}{2} \int_{-\infty}^{+\infty} \Xi(u) \; du \; .$$

$\mathbf{\Sigma^+ - \Sigma^-}$ yields **permanent displacement** of test masses. Non-linear effect, i.e. nonlinear memory effect.

$$\Sigma(u) \; = \; \Sigma^- \; + \; \frac{1}{2} \int_{-\infty}^{u} \Xi(u') \; du'$$

$\mathbf{\Sigma(u) - \Sigma^-}$ related to **instantaneous displacements** of faraway test masses w.r.t. reference test mass, relative to which they are initially at rest.

Ξ is dimensionless.

Σ has dimensions of length.

F has dimensions of length.

In the example of a **binary coalescence**

=> The **solution** formula for $(\Sigma^+ - \Sigma^-)$ from above

- Has a **nonlinear contribution from** F

  and

- A **linear contribution** from

$$(P - P_{[1]})^+ - (P - P_{[1]})^-$$

Linear effect

=> was known for a long time in the slow motion limit

[Ya.B. Zel'dovich, A.G. Polnarev 1974]

Nonlinear effect

=> was found by **[D. Christodoulou 1991]**.

Contribution from electro-magnetic field to nonlinear effect

=> was found by **[L. Bieri, P. Chen, S.-T. Yau 2010 and 2011]**.

Contribution from neutrino radiation to nonlinear effect

=> recent result by **[L. Bieri, D. Garfinkle 2012 and 2013]**.

Open Questions

Many.....

For instance

- Geometry and null asymptotics of other spacetimes?

- What are the patterns in the gravitational radiation for various astrophysical scenarios? How is the geometry changed?

- What happens, when inserting other fields on the right hand side of Einstein equations?