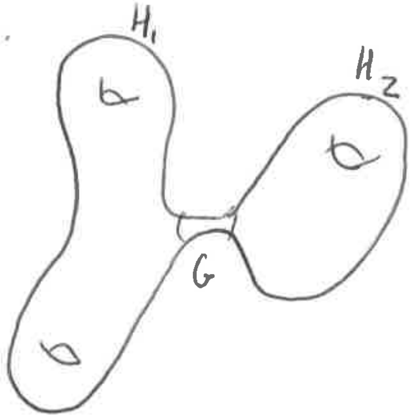
Ringström - Cosmology

Fig 1.



COSMOLOGY

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Copernican principle: We do not occupy a privileged position in the universe.

Cosmological principle: The universe is spatially homogeneous and isotropic (on any given slice of a foliation of the universe). This leaves us with either a spherical, flat or hyperbolic universe and only one function of time scaling.

Is the universe static or expanding or contracting? Einstein first wrote down that it was static. He later regretted that. One problem with a static universe is Olber's paradox: If we live in a static universe, then the brightness of light from the sky should be infinite in any direction.

More modernly, we have noticed that most stars are redshifted, and so they are moving away. Later, Lemaitre and Hubble correlated redshift and distance. They realized that the further away the stars are, the faster they're moving away. This implies that we have an expanding universe. This suggests that there was some kind of big bang in the past.

This was corroborated by the cosmic microwave background radiation.

The biggest shift was in 98-99. Type 1a Supernovae (SNIa) suggest that the universe is expanding at an accelerating rate. Usually this is modeled by a positive cosmological constant. There are lots of people working on this now.

Questions:

(1) Stability? The solutions used in cosmology are highly symmetric. They may or may not be good models. For instance, the Einstein static model is not stable.
(2) Singularities? Einstein did not like these singularities in the highly symmetric solutions of his day. He though that if we perturb them, maybe they won't show up. But we now know that curvature singularities, etc. often show up. Are singularities generic in some sense?
(3) Determinism (via strong cosmic censorship)? It fails sometimes, but the known solutions are special.
(4) Homogenization/isotropization. It would be preferable to derive this as a result of the evolution instead of putting it in by hand.

Methods:

(1) Finding explicit solutions.
(2) Lorentz geometric methods, such as Hawking-Penrose incompleteness theorems. Generically, spacetimes have singularities in the sense of causal geodesic incompleteness.

(3) Cauchy problem, i.e. considering the equations as an initial value problem. The stability problem is really here.
(4) Constructing solutions with prescribed asymptotics.

Cauchy problem: Most results that have been done fall into one of two categories: stability results, and results concerning symmetric situations. And the stability results are really for highly symmetric situations anyway. This is a severe topological restriction. We throw out most of Einstein's theory before we've even started.

Expanding direction in the vacuum setting: Fisher-Moncrief and Anderson have made some conjectures that are more general. Our setup will be (M, g), a vacuum spacetime, satisfying the Einstein equations and we'll assume it's foliated by constant mean curvature (CMC) compact Cauchy hypersurfaces, which is a restriction. We don't want zero mean curvature to be attained here, or else we would have future AND past incomplete geodesics by Hawking incompleteness. Let Σ_τ be the hypersurface of CMC τ.

There exists a $\tau_0 > 0$ such that $J^+(\Sigma_{\tau_0}$ is foliated by CMC hypersurfaces exhausting the range $[\tau_0, 0)$. We are also assuming that (M, g) is future causally geodesically complete.

What can we say about the future asymptotics? Fisher-Moncrief: The reduced Hamiltonian $H_{red} = |\tau^3|\text{vol}\Sigma_\tau$ decays.

Michael Anderson: $\text{vol}\Sigma_\tau/t_\tau^3$ is also decaying, where t_τ is the Lorentzian distance from Σ_{τ_0} to Σ_τ.

Suppose these quantities are constant on an interval (τ_1, τ_2). Then the solution is $g = -dt^2 + t^2 g_H$, which is really Minkowski space in disguise. This is called the Milne model. This is a natural fixed point of the flow. Is it an attractor?

There is a relationship between the Reduced Hamiltonian and the sigma constant, so you might hope it approaches it in some sense.

Consider a rescaling of $\hat{g} = \tau^2 g_\tau$ (Fisher-Moncrief) or $\tilde{g}_\tau = \frac{1}{t_\tau^2} g_\tau$ (Andersson). We can think of these as a family of metrics on a fixed closed 3-manifold. The idea then is then that $\Sigma = G \cup H$, where G is a collection of graph manifolds and H a collection of complete hyperbolic manifolds. The union is along 2-tori. See figure 1. We expect that it should collapse on G and expand on H_i as we evolve, and so we get (local) homogenization and isotropization on H_i

Is there any reason to believe this picture? What is the support? It's clear that the hyperbolic pieces are of central importance for this model. Thus we need to look at the Milne model first. The Milne model is stable (Andersson-Moncrief). It's an attractor in some sense.

There are also studies of the $U(1)$ problem. Suppose we have a spacetime $\mathbb{R} \times S^1 \times \Sigma_k$, where the last is a higher genus surface, looking at the stuff that is $U(1)$ symmetric. This works out nicely. This is in G, and we get the collapse we expect. (Due to Choquet-Bruhat and Moncrief.)

We'd like to get confirmation when there's some kind of non-trivial decomposition. There are results of this type, but they're based on making a priori assumptions. Given a priori assumptions (concerning the curvature of these manifolds, etc.) we get the desired conclusions. (Due to M. Anderson and Reiris.) Of course it's not clear these are applicable.

Suppose we look at the expanding direction, where there is a positive cosmological constant. Wald: Look at Einstein field equations with a positive cosmological constant $\Lambda > 0$. Assume that the matter (not Λ) satisfies the strong and dominant energy conditions (SEC and DEC respectively). Solutions (which correspond to left invariant i.d. on 3 dimensional Lie groups G) exist globally to the future. (Let's throw out the Minkowski-Sachs solution.) In this setup, the Hamiltonian constraint will say that

$$|\mathrm{tr} k|^2 = -\frac{3}{2}\bar{S} + \frac{3}{2}|\sigma|^2 + 3\Lambda + 3\rho.$$

The only term here that could be negative is the scalar curvature S.

If the universal covering group of G is not isometric to $SU(2)$, then $\bar{S} \leq 0$. Let's change curvature convention. Modulo that, then $\mathrm{tr} k \geq 3H$ where $H = \sqrt{\Lambda/3}$.

Let $X = (\mathrm{tr} k)^2 - 3\Lambda$. Then $X \geq 0$ and $\dot{X} \leq -\frac{2}{3}(\mathrm{tr} k)X$ and so $X \to 0$ exponentially. Thus $\bar{S}$, σ and ρ all must go to zero exponentially. Thus we expect we get isotropization, since the shear is going to zero.

This is very different from the vacuum behavior. Here, because $\Lambda > 0$, this is very robust and stable.

Cosmic no-hair conjecture: For generic initial data, leading to future complete solutions with $\Lambda > 0$, then late time observers will consider the solution to approach de Sitter space, $-dt^2 + \cosh^2 t\, g_{S^3}$.

There are plenty of stability results that seem to confirm this for many matter models.

The case with positive cosmological constant is much more stable as compared to the vacuum case.

Construction solutions with prescribed asymptotics: We can't do this in general, as it requires restrictions. However, given suitable symmetry assumptions, dimensional assumptions and/or matter model assumptions, you can construct such solutions.

We will only discuss one such solution, by Lars Andersson and Rendall. Consider the Einstein field equations (EFE) with scalar field (or stiff fluid). We can write down the equations in Gaussian coordinates, $-dt^2 + g_{ij} dx^i \otimes dx^j$.

They essentially throw out the spatial derivatives, then they end up with something they call a velocity dominated (VD) system. Given the VD solution, there exists a unique solution to the full EFE with the relevant matter model that has the prescribed VD asymptotics.

There are some drawbacks, of course. First, it is in the real analytic setting (which is an unusual case for GR). Next, this need not correspond to an open set of regular initial data. There are results to remove the real analytic setting restriction. The other issue is not so clear, but there has been some progress by Rodnianski and Speck (sp?).