Piotr Bizon

Fig 1

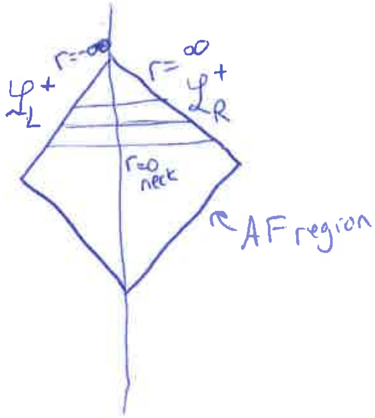


Fig 2

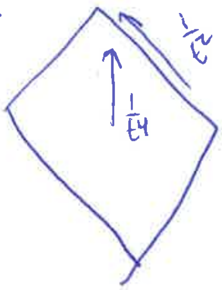
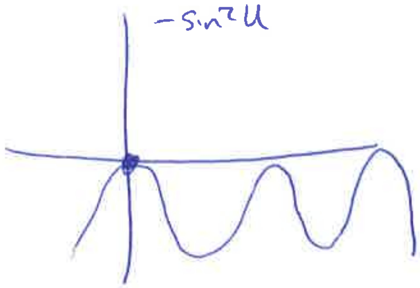


Fig 3



~~Fig 4~~

SIMPLE NONLINEAR WAVES ON CURVED MANIFOLDS

PIOTR BIZON

The models: See figure 1. For the wormhole case, there are 2 ends, connected by a wormhole region near $r = 0$.

Minkowski background: The wave map decay rate is not rigorously proven since the WM equations are not conformally covariant, and so we can't use the conformal method. Also, the blow up is stable in the perturbative sense.

Existence and (in)stability of static solutions: There is a degeneracy in the vacuum case, at $W = \pm 1$. Solutions are attracted to one or the other, but in the middle, there must be a fixed point/solution. These are unstable static solutions. (But $W \equiv 0$ is stable).

Question: Can you use this methods to simplify the proof of Bartnik Mckeen? Not really, because you need a flow, and it is not known what flow to work. In theory, it does work.

Hyperboloidal initial value problem: See figure 1. Lines of constant s are horizontal in that picture.

Relaxation to equilibrium (heuristic picture): Analyticity is an unusual requirement, and some people don't like it, but it replaces the condition of "outgoing waves" in physics. Analyticity here means we have it even at $y = \pm\pi/2$ which is a singular point of the ODE.

Heteroclinic connection between W_1 and the vacuum: The shown graph is for initial data is very close to $W(0, y) = y + \epsilon$, $W_t(0, y) = 0$. First, it decays along quasinormal modes. Then, because it is unstable, the instability becomes important and it goes up. It then begins to decay.

Tail at Scri: See figure 2. The slide shows the first 4 derivatives. This behavior is because $1 - w^2 = \frac{r^2}{u^2(u+2r)^2} = \frac{1}{u^2}\frac{1}{(u\rho+2)^2}$, using $\rho = 1/r$, $u = t - 4$ and $v = t + r$.

Wave maps on the wormhole: Solutions are monotone. $x = \sinh^{-1}(r/a)$, $U'' + \tanh(x)U' - \sin(2U) = 0$. Here, $\sinh^{-1}$ is a friction term. See figure 3. If it is given enough velocity will reach somewhere on the curve in infinite time.

Weakly turbulent instability of the ground state: Solutions are smooth for all time, but not analytic. The coefficients are $a_n(t) \sim e^{-\rho(t)n}$ and $n \to \infty$, where $\rho(t)$ is the radius of analyticity. If it goes to zero, the solutions get rougher and rougher (though smooth at any time). This happens for "generic" data, but not always. There are "islands of stability."

Energy spectrum for the two-mode perturbations of W_0: Mode number is on right hand side. Conjecture: the ρ goes to 0 at infinity.

Sobolev norms for two-mode perturbations of W_0: Right hand side is the subscript for the Sobolev norms.

Sobolev norms for Gaussian perturbations of W^1: Same picture, but note that the linearized spectrum is not resonant. Thus, resonance is probably a driving force for weak turbulence.

Simple nonlinear waves on curved manifolds

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joint work with Michał Kahl and Maciej Maliborski

Berkeley, 18 November 2013

Motivation

- Physical motivation: many physical theories involve nonlinear wave equations on curved manifolds
- Mathematical motivation: by changing the domain of a nonlinear wave equation from the flat to a curved manifold one can design an equation with some desired properties (”designer” PDEs)
- We want to design simplest possible settings for studying the following two phenomena for nonlinear dispersive wave equations:
  - **Relaxation to a (nontrivial) equilibrium** on an unbounded domain through dissipation by dispersion
  - **Weak turbulence** on a bounded domain - shift of the energy spectrum from low to high frequencies
- Warning: I will speak the language of theoretical physics

The models

- Domain: $3+1$ dimensional manifold with ultrastatic spherical metric

$$g = -dt^2 + dr^2 + R^2(r)\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

- Equations: equivariant SU(2) Yang-Mills and wave maps into S^3

$$W_{tt} = W_{rr} + \frac{W(1-W^2)}{R^2} \tag{YM}$$

$$U_{tt} = U_{rr} + \frac{2R'}{R^2}\,U_r - \frac{\sin(2U)}{R^2} \tag{WM}$$

- I will discuss three cases:
  - $R = r$ $\qquad r \geq 0$ $\qquad$ Minkowski
  - $R = \sqrt{r^2 + a^2}$ $\qquad -\infty < r < \infty$ $\qquad$ **wormhole**
  - $R = \sin r$ $\qquad 0 \leq r \leq \pi$ $\qquad$ **Einstein's static universe**

Minkowski background

- Invariance under dilation $(t,r) \to (t/\lambda, r/\lambda)$. Energy $E \to \lambda^{\alpha} E$ where $\alpha = -1$ for YM (subcritical) and $\alpha = 1$ for WM (supercritical)
- $\Rightarrow$ no static solutions with finite nonzero energy
- YM: global-in-time existence for any smooth initial data and asymptotic decay to vacuum with the sharp decay rate

$$1 - W^2 \sim \frac{r^2}{\langle t-r\rangle^2 \langle t+r\rangle^2} \quad \text{for} \quad t \to \infty$$

- WM: a) small data: global-in-time existence and asymptotic decay to vacuum with the decay rate

$$U \sim \frac{rt}{\langle t-r\rangle^3 \langle t+r\rangle^3} \quad \text{for} \quad t \to \infty$$

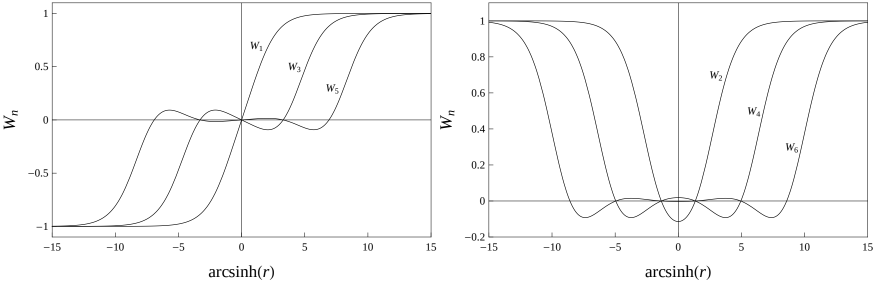
  b) large data: self-similar blowup in time $T < \infty$

$$U(t,r) \to S\left(\frac{r}{T-t}\right) \quad \text{as} \quad t \nearrow T$$

Yang-Mills on the wormhole

$$W_{tt} = W_{rr} + \frac{W(1 - W^2)}{r^2 + a^2}$$

- The length scale a plays two roles:
  - breaks scale invariance
  - removes the singularity at $r = 0 \Rightarrow$ global-in-time regularity
- However, not all solutions decay to vacuum because ...
- there exist infinitely many smooth finite energy static solutions $W_n(r)$



Existence and (in)stability of static solutions

Two ways of proving the existence of static solutions $W_n(r)$:

- ODE shooting method. Key to the proof is the fact that in terms of the variable $x = \operatorname{arcsinh}(r/a)$ the time-independent equation $W'' - \tanh x\, W' + W(1 - W^2) = 0$ is asymptotically autonomous.
- Morse-theoretic argument of Corlette and Wald for the heat flow $W_t = W_{rr} + \frac{W(1-W^2)}{r^2+a^2}$. Key to the proof is the global regularity of the flow and the reflection symmetry $W \to -W$ (with a fixed point $W = 0$ of infinite index).
- The advantage of the heat flow method is its wide applicability (for example to black hole backgrounds) and strong implications regarding stability properties of static solutions. This method gives for free that the solution W_n has n unstable directions (i.e. the linearized operator around W_n has n negative eigenvalues).

Hyperboloidal initial value problem

- The hyperboloidal formulation provides a very natural setting for studying the relaxation to equilibrium due to dispersion of waves to infinity (see Anıl's talk tomorrow)
- We define new variables ($-\infty < s < \infty$, $-\pi/2 < y < \pi/2$)

$$s = t - \sqrt{r^2 + a^2}, \quad y = \arctan(r/a)$$

- Then the YM equation becomes

$$W_{ss} + 2\sin y\, W_{sy} + \cos y\, W_s = \partial_y \left(\cos^2 y\, W_y\right) + W(1 - W^2)$$

- The Bondi-type energy

$$\mathscr{E} = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(W_s^2 + \cos^2 y\, W_y^2 + \frac{1}{2}(1 - W^2)^2 \right) dy$$

decreases due the flux of energy through Scri

$$\frac{d\mathscr{E}}{ds} = -W_s^2(s, \frac{\pi}{2}) - W_s^2(s, -\frac{\pi}{2})$$

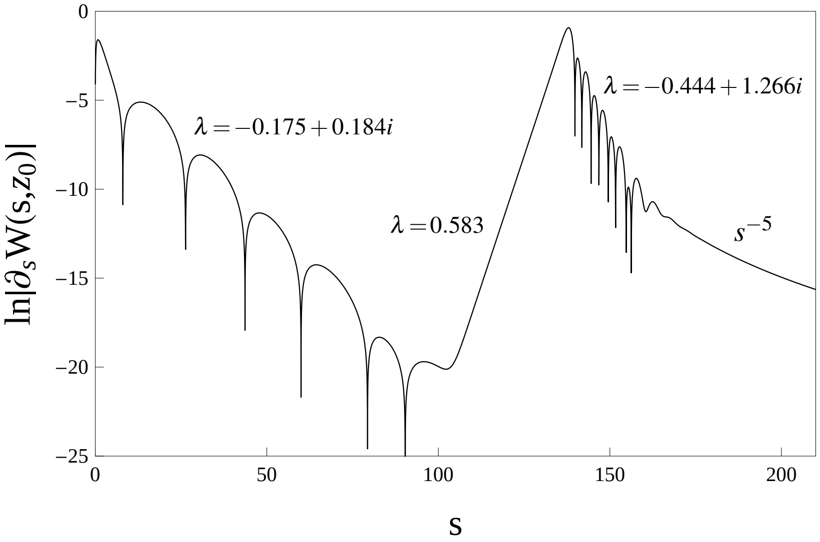
Relaxation to equilibrium (heuristic picture)

- As $s \to \infty$, the energy $\mathscr{E}(s)$ tends to a nonnegative limit which is equal to the energy of one of the static solutions W_n. Since the solution W_n has n instabilities, generic initial data evolve to $W_0 = 1$ (or $-W_0$) but the solutions W_n $(n \geq 1)$ can appear as codimension-n attractors (in particular, W_1 is an attractor for odd initial data).
- What is the rate of convergence to W_n?
- **Quasinormal ringdown** for intermediate times. The linearized flow around W_n leads to the eigenvalue problem $(W(s,y) = W_n + e^{\lambda s} v(y))$

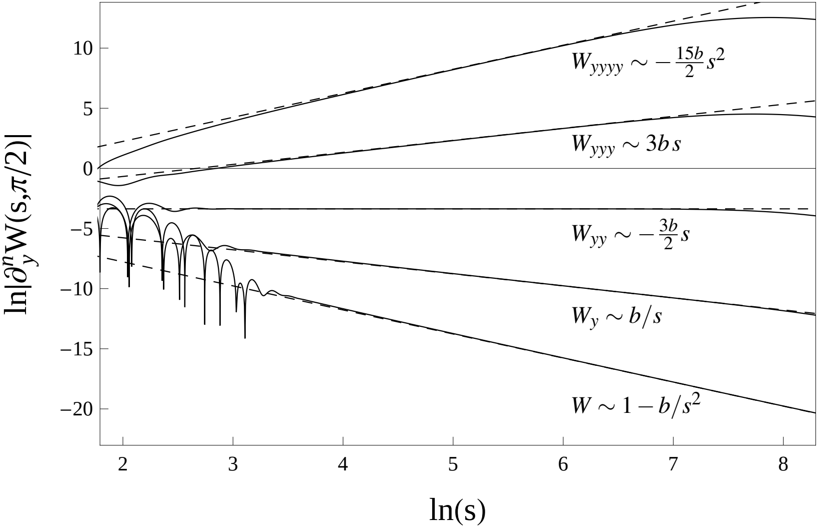
$$\lambda^2 v + \lambda \left(\cos y\, v + 2 \sin y\, v_y\right) - \partial_y \left(\cos^2 y\, v_y\right) + (3W_n^2 - 1)\, v = 0$$

  Quasinormal modes are defined as *analytic* eigenmodes with $\Re(\lambda) < 0$.
- **Nonlinear tail** for late times. The same decay (including Scri) as in flat space (derived by the formal nonlinear perturbation analysis and verified by numerics). A linear tail due to backscattering is subdominant.

Heteroclinic connection between W_1 and the vacuum



Tail at Scri

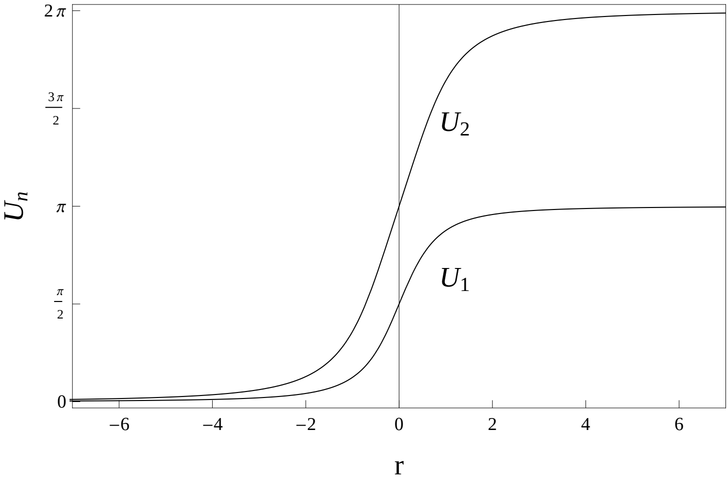

$W_{yyyy} \sim -\frac{15b}{2} s^2$
$W_{yyy} \sim 3b\,s$
$W_{yy} \sim -\frac{3b}{2} s$
$W_y \sim b/s$
$W \sim 1 - b/s^2$
10
5
0
−5
−10
−15
−20
2
3
4
5
6
7
8
$\ln|\partial_y^n W(s,\pi/2)|$
ln(s)

Wave maps on the wormhole

$$U_{tt} = U_{rr} + \frac{2r}{r^2+a^2} U_r - \frac{\sin(2U)}{r^2+a^2}$$

- The length scale a removes the singularity at $r=0$ and renders the WM equation subcritical $\Rightarrow$ global-in-time regularity.
- Finiteness of energy requires that $U(t,-\infty) = m\pi$, $U(t,\infty) = n\pi$. We choose $m=0$ so n determines the topological degree of the map (which is preserved in evolution).
- For each n there exists a unique static solution $U_n(r)$ (harmonic map) which minimizes the energy in its class (proof: shooting or heat flow).
- The harmonic maps U_n are linearly stable. Proof: Direct calculation shows that $v_n = \sqrt{r^2+a^2}\, U_n'(r)$ is the zero mode of the operator $L - a^2/(r^2+a^2)^2$, where L is the linearized operator around U_n. Since $U_n(r)$ is monotone increasing, the zero mode $v_n(r)$ has no nodes and the claim follows from Sturm's oscillation theorem.

The harmonic maps of degree one and two.



Soliton resolution conjecture for WM on the wormhole

Conjecture

For any smooth initial data of degree n there exists a unique and smooth global solution which converges asymptotically to the harmonic map U_n.

- Recently, an analogous result was proved for equivariant wave maps exterior to a ball by Kenig, Lawrie, and Schlag. It is likely that their proof can be adopted to our case.
- What is the rate of convergence to U_n?
- The hyperboloidal formulation

$$U_{ss} + 2\sin y\, U_{sy} + \frac{1+\sin^2 y}{\cos y}\, U_s = \cos^2 y\, U_y - \sin(2U)$$

  leads to similar conclusions as for YM.
- This is an ideal setting for developing insight into a relationship between topological and nonlinear stability.

Yang-Mills on the Einstein universe

$$W_{tt} = W_{rr} + \frac{W(1-W^2)}{\sin^2 r}$$

- Remark: Yang-Mills equations in four dimensions are conformally invariant so restricting the domain to the northern hemisphere we get YM on Anti-de Sitter. One can enforce the Dirichlet/Neumann boundary condition on the equator (conformal boundary of AdS) by prescribing odd/even initial data on S^3.
- Global-in-time existence for any smooth initial data
- Regularity requires that $W(t,0) = \pm 1$ and $W(t,\pi) = \pm 1$ $\Rightarrow$ two topological sectors $N = 0, 1$.
- In each sector there is a unique ground state static solution: $W_0 = 1$ (vacuum) and $W_1 = \cos(r)$ (kink). **Are they stable?**
- The domain is compact so dissipation by dispersion is absent and therefore asymptotic stability of static solutions is excluded.

Spectrum of linearized perturbations

- Linearized perturbations $v = W - W_N$ around the static solutions satisfy

$$v_{tt} + Lv = 0\,, \qquad L = -\frac{d^2}{dr^2} + \frac{3W_N^2 - 1}{\sin^2 r}$$

The linear operator L is essentially self-adjoint on $L^2([0,\pi], dr)$.

- The eigenvalues are $(n = 0, 1, ...)$

$$\omega_n^2 = (2+n)^2 \quad \text{for} \quad N=0 \quad \text{and} \quad \omega_n^2 = (2+n)^2 - 3 \quad \text{for} \quad N=1$$

- The corresponding eigenfunctions are

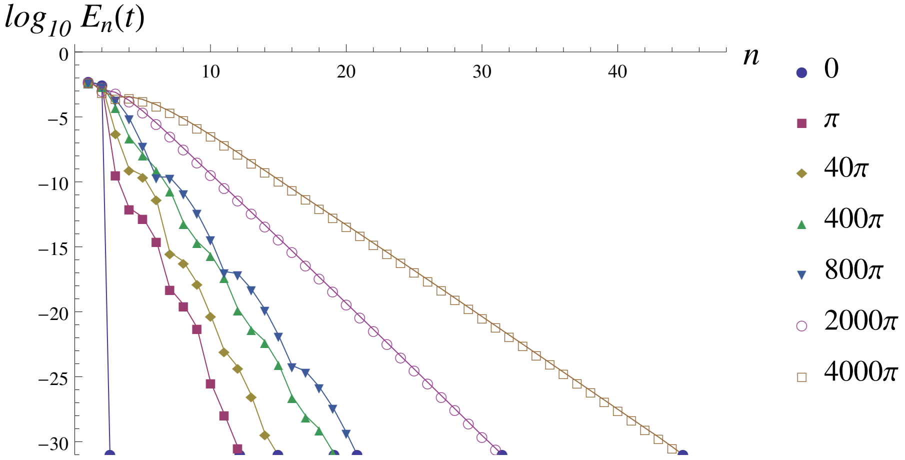
$$e_n(r) = c_n\,(\cos r \sin(2+n)r - (2+n)\cos(2+n)r)$$

where c_n is a normalization factor.

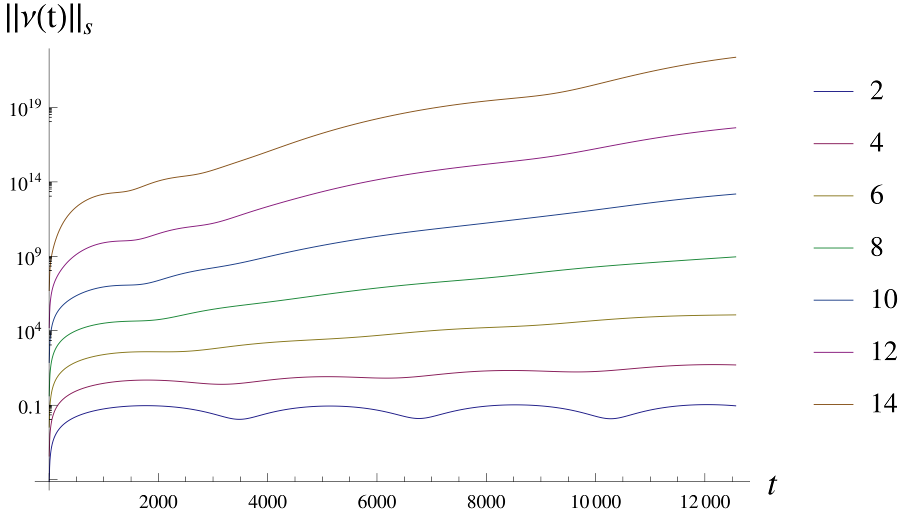
Weakly turbulent instability of the ground state

- Substituting $W(t,r) = W_N + \sum a_n(t)e_n(r)$ into the YM equation and projecting on the basis e_n we get an infinite system of coupled nonlinear oscillators
$$\ddot{a}_n + \omega_n^2 a_n = \sum b_{jk}^n a_j a_k + \sum c_{jkm}^n a_j a_k a_m$$
- This system is **fully resonant** for $N=0$ and non-resonant for $N=1$.
- The transfer of energy between the modes can be quantified by the energy spectrum $E_n(t) = \dot{a}_n^2 + \omega_n^2 a_n^2$ or the norms $\|v(t)\|_s = \left(\sum \omega_n^{2s} a_n^2\right)^{1/2}$.
- Numerical simulations of generic small perturbations of W_0 indicate that
  - the norms $\|v(t)\|_s$ (for $s \geq 2$) tend to grow unboundedly in time
  - the slope of $\log E_n$ flattens with time (i.e. the radius of analyticity shrinks)
- This means that energy is being transferred from low to high frequencies. Such a shift of energy spectrum is usually called **weak turbulence**.
- There exist special initial data which give rise to **time-periodic solutions**. The neighbouring solutions do not seem to exhibit the turbulent behavior – islands of (meta)-stability.

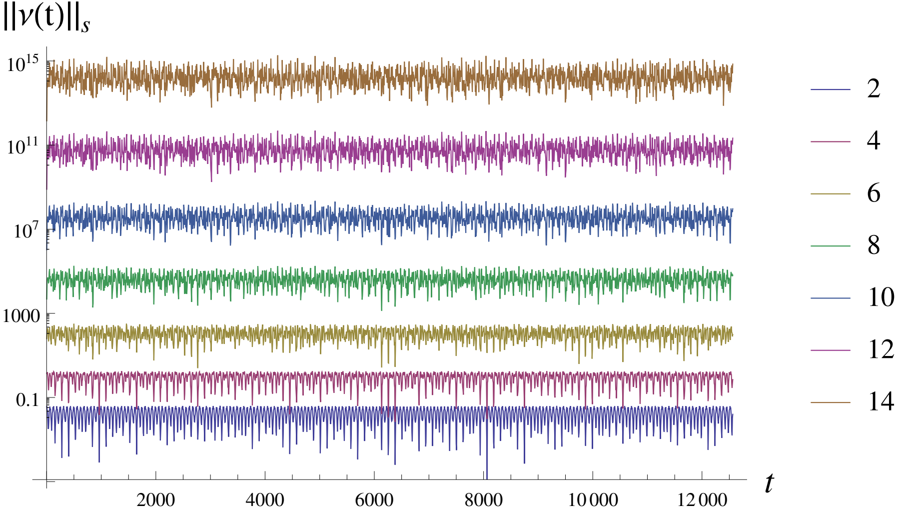
Energy spectrum for two-mode perturbations of W_0



Sobolev norms for two-mode perturbations of W_0



Sobolev norms for Gaussian perturbations of W_1



Wave maps on the Einstein universe

$$U_{tt} = U_{rr} + 2\cot r\, U_r - \frac{\sin(2U)}{\sin^2 r}$$

- Infinitely many static solutions (harmonic maps between 3-spheres)
- Blowup for large data is governed by self-similar wave maps from Minkowski into S^3 (blowup does not see the curvature)
- The basic question: do small enough smooth perturbations of the zero solution remain smooth forever? In other words: **is there a nonzero threshold for blowup**?
- One may speculate that the weakly turbulent behavior combined with the supercritical scaling of energy may lead to blowup for arbitrarily small perturbations (as has been recently found for perturbations of AdS)
- At the moment the numerical evidence is not conclusive

Final remarks

- Playing with the domains of nonlinear wave equations one can construct simple toy models for studying interesting physical phenomena
- Many statements in this talk are not rigorously proved and should be treated as (plausible) conjectures. I hope some of them are within the reach of current PDE technology.
- One has to remember that the road from the physical truth to the mathematical truth may be long and full of obstacles.