Linear Stability of the Schwarzschild Solution

(joint with M. Dafermos and I. Rodnianski)

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Overview

1. Statement of the problem and the result
2. The linearized equations
3. What was known before
4. Estimates for the linearized system
5. Outlook on non-linear problems

The Black Hole Stability Problem...

... was introduced in previous talks. It corresponds to understanding the dynamics of the vacuum Einstein equations

$$R_{\mu\nu}\left[g\right] = 0 \tag{1}$$

in a neighborhood of the the Kerr family of solutions $(\mathcal{M}, g_{M,a})$.

An obvious approach is to try to linearize (1). In harmonic gauge

$$\Box_g g_{\mu\nu} = Q\left(\partial g, \partial g\right)$$

so simplest toy-model is $\Box_{g_{M,a}}\psi = 0$. This problem is now understood.

[Dafermos-Rodnianski, Andersson-Blue, Tataru-Tohaneanu, Aretakis, Shlapentokh-Rothman, Luk, ...].

Theorem [Dafermos, G.H., Rodnianski]

The Schwarzschild spacetime is linearly stable against gravitational perturbations.

The Einstein equations and Linearization

The analytical content of $R_{\mu\nu}[g] = 0$ is contained in

$$\nabla^{\mu} W_{\mu\nu\sigma\tau} = 0 \qquad \text{Bianchi Equations}$$

$$\nabla\Gamma + \Gamma\Gamma = W \qquad \text{Structure Equations}$$

Linearize around background with connection Γ_0 and curvature $W_\circ$:

$$\left(\partial + \Gamma_\circ + \Gamma^{(1)}\right)\left(W_\circ + W^{(1)}\right) = 0$$

$$\left(\partial + \Gamma_\circ + \Gamma^{(1)}\right)\left(\Gamma_\circ + \Gamma^{(1)}\right) = W_\circ + W^{(1)}$$

$$\left(\partial + \Gamma_\circ + \Gamma^{(1)}\right)\left(W_\circ + W^{(1)}\right) = 0 \tag{2}$$

$$\left(\partial + \Gamma_\circ + \Gamma^{(1)}\right)\left(\Gamma_\circ + \Gamma^{(1)}\right) = W_\circ + W^{(1)} \tag{3}$$

Around Minkowski, $W_\circ = 0$ so $\Gamma^{(1)}$, drops out of (2). $\rightarrow$ Decoupling!

The linearized spin 2 equations $\partial^\mu W_{\mu\nu\sigma\tau} = 0$ on Minkowski were understood in [Chr-Kl] paper "Asymptotic Properties of linear field equations in Minkowski space".

$\rightarrow$ Decay via conformal vectorfields and Bel-Robinson tensor for W

Around Schwarzschild $W_\circ \neq 0$, so **coupling even at linear level**:

$$\begin{aligned}(\partial + \Gamma_\circ)\, W^{(1)} + W_\circ \Gamma^{(1)} &= 0 \\ (\partial + \Gamma_\circ)\, \Gamma^{(1)} + \Gamma_\circ \Gamma^{(1)} &= W^{(1)}\end{aligned} \tag{4}$$

1. How does one prove boundedness for the coupled system (4)?
2. Do there exist quantities (components of $W^{(1)}$ or $\Gamma^{(1)}$) which
   (a) decouple
   (b) satisfy a "good" equation
   (c) control the remaining quantities

The study of (4) has a long tradition in the physics literature (cf. the monograph [Chandrasekhar]). However, a uniform boundedness statement was never obtained.

A fruitful way to study (4) proceeds via a null-decomposition. Suppose we look at a family of metrics in double null coordinates [Christodoulou]

$$g = -\Omega^2 du dv + \not{g}_{AB} \left(d\theta^A + b^A dv\right) \left(d\theta^A + b^A dv\right) \tag{5}$$

Associated null-frame

$$e_3 = \frac{1}{\Omega}\partial_u \quad , \quad e_4 = \frac{1}{\Omega}\left(\partial_v + b^A e_A\right) \quad , \quad e_A = \frac{\partial}{\partial\theta^A} \tag{6}$$

$\rightarrow$ Express the equations with respect to this frame and linearize.

Canonical Notation

There is a canonical notation.

$$\chi_{AB} = g\left(\nabla_{e_A} e_4, e_B\right) \quad \text{second fundamental form of } S^2_{u,v} \text{ in } C_u \tag{7}$$

Construct $tr\chi$ and $\hat{\chi}$, which is symmetric traceless.
In Schwarzschild, we have $\hat{\chi} = 0$ and $\Omega tr\chi = \frac{2}{r}\left(1 - \frac{2M}{r}\right)$.
Hence write $\hat{\chi}$ and $\left(\Omega tr\chi\right)^{(1)}$ for linearized part.

For the curvature components,

$$\alpha_{AB} = W\left(e_4, e_A, e_4, e_B\right) \quad \text{and} \quad \underline{\alpha}_{AB} = W\left(e_3, e_A, e_3, e_B\right) \tag{8}$$

which vanish in Schwarzschild. Also, $\rho = W\left(e_3, e_4, e_3, e_4\right)$.
In Schwarzschild $\rho = -\frac{2M}{r^3}$ and hence write $\rho^{(1)}$ for linearized part.

$$\underline{D}_S \frac{\sqrt{g}^{(1)}}{\sqrt{g}_S} = \left(\Omega tr\underline{\chi}\right)^{(1)}$$

$$D_S \frac{\sqrt{g}^{(1)}}{\sqrt{g}_S} = (\Omega tr\chi)^{(1)} - d\not{i}vb$$

$$\frac{\partial}{\partial u} b^A = 2\Omega_S^2 \left[\left(\eta - \underline{\eta}\right)^\sharp\right]^A$$

$$D_S \left(\Omega tr\underline{\chi}\right)^{(1)} = \Omega_S^2 \left(2d\not{i}v\underline{\eta} + 2\hat{\rho} + 4\rho_S \frac{\Omega^{(1)}}{\Omega_S}\right) - \frac{1}{2} (\Omega tr\chi)_S \left(\left(\Omega tr\underline{\chi}\right)^{(1)} - (\Omega tr\chi)^{(1)}\right)$$

$$\underline{D}_S (\Omega tr\chi)^{(1)} = \Omega_S^2 \left(2d\not{i}v\eta + 2\hat{\rho} + 4\rho_S \frac{\Omega^{(1)}}{\Omega_S}\right) - \frac{1}{2} (\Omega tr\chi)_S \left(\left(\Omega tr\underline{\chi}\right)^{(1)} - (\Omega tr\chi)^{(1)}\right)$$

$$D_S (\Omega tr\chi)^{(1)} = -(\Omega tr\chi)_S (\Omega tr\chi)^{(1)} + 2\omega_S (\Omega tr\chi)^{(1)} + 2 (\Omega tr\chi)_S \, \omega^{(1)}$$

$$\underline{D}_S \left(\Omega tr\underline{\chi}\right)^{(1)} = -\left(\Omega tr\underline{\chi}\right)_S \left(\Omega tr\underline{\chi}\right)^{(1)} + 2\underline{\omega}_S \left(\Omega tr\underline{\chi}\right)^{(1)} + 2 \left(\Omega tr\underline{\chi}\right)_S \underline{\omega}^{(1)}$$

$$\nabla\!\!\!\!/_3\left(\frac{1}{\Omega}\underline{\hat{\chi}}\right)+\frac{1}{\Omega}tr\underline{\chi}\underline{\hat{\chi}}=-\frac{1}{\Omega}\underline{\alpha}\quad,\quad\nabla\!\!\!\!/_4\left(\frac{1}{\Omega}\hat{\chi}\right)+\frac{1}{\Omega}tr\chi\hat{\chi}=-\frac{1}{\Omega}\alpha$$

$$\nabla\!\!\!\!/_3\left(\Omega\hat{\chi}\right)+\frac{1}{2}\Omega tr\underline{\chi}\hat{\chi}+\frac{1}{2}\Omega tr\chi\underline{\hat{\chi}}=-2\Omega\mathcal{D}\!\!\!\!/_2^{\star}\eta$$

$$\nabla\!\!\!\!/_4\left(\Omega\underline{\hat{\chi}}\right)+\frac{1}{2}\Omega tr\chi\underline{\hat{\chi}}+\frac{1}{2}\Omega tr\underline{\chi}\hat{\chi}=2\Omega\mathcal{D}\!\!\!\!/_2^{\star}\eta\,.$$

$$d\!\!\!/iv\,\underline{\hat{\chi}}=-\frac{1}{2}\eta tr\underline{\chi}+\underline{\beta}+\frac{1}{2\Omega}\nabla\!\!\!\!/_A\left(\Omega tr\underline{\chi}\right)^{(1)}$$

$$d\!\!\!/iv\,\hat{\chi}=-\frac{1}{2}\underline{\eta}tr\chi-\beta+\frac{1}{2\Omega}\nabla\!\!\!\!/_A\left(\Omega tr\chi\right)^{(1)}\quad.$$

$$\nabla\!\!\!\!/_3\underline{\eta}=\frac{1}{2}tr\underline{\chi}\left(\eta-\underline{\eta}\right)+\underline{\beta}\quad,\quad\nabla\!\!\!\!/_4\eta=-\frac{1}{2}tr\chi\left(\eta-\underline{\eta}\right)-\beta$$

$$D_S\underline{\omega}^{(1)}=-\Omega_S^2\left(\rho^{(1)}+2\rho_S\frac{\Omega^{(1)}}{\Omega_S}\right)\quad,\quad\underline{D}_S\omega^{(1)}=-\Omega_S^2\left(\rho^{(1)}+2\rho_S\frac{\Omega^{(1)}}{\Omega_S}\right)$$

$$\omega^{(1)}=D_S\left(\frac{\Omega^{(1)}}{\Omega_S}\right)\ ,\ \underline{\omega}^{(1)}=\underline{D}_S\left(\frac{\Omega^{(1)}}{\Omega_S}\right)\ ,\ \left(\eta+\underline{\eta}\right)^{(1)}=2\nabla\!\!\!\!/_A\left(\frac{\Omega^{(1)}}{\Omega_S}\right)$$

$$\nabla\!\!\!\!/_3 \alpha + \frac{1}{2} tr\underline{\chi}\alpha + 2\underline{\hat{\omega}}\alpha = -2\mathcal{D}\!\!\!\!/_2^\star \beta - 3\hat{\chi}\rho_0$$

$$\nabla\!\!\!\!/_4 \beta + 2tr\chi\beta - \hat{\omega}\beta = d\not{i}v\,\alpha$$

$$\nabla\!\!\!\!/_3 \beta + tr\underline{\chi}\beta + \underline{\hat{\omega}}\beta = \mathcal{D}\!\!\!\!/_1^\star \left(-\rho^{(1)}, \sigma\right) + 3\eta\rho_0$$

$$\nabla\!\!\!\!/_4 \rho^{(1)} + \frac{3}{2} tr\chi\rho^{(1)} = d\not{i}v\,\beta - \frac{3}{2}\frac{\rho_S}{\Omega_S}\,(\Omega tr\chi)^{(1)}$$

$$\nabla\!\!\!\!/_3 \rho^{(1)} + \frac{3}{2} tr\underline{\chi}\rho^{(1)} = -d\not{i}v\,\underline{\beta} - \frac{3}{2}\frac{\rho_S}{\Omega_S}\left(\Omega tr\underline{\chi}\right)^{(1)}$$

$$\nabla\!\!\!\!/_4 \sigma + \frac{3}{2} tr\chi\sigma = -c\not{u}rl\,\beta$$

$$\nabla\!\!\!\!/_3 \sigma + \frac{3}{2} tr\underline{\chi}\sigma = -c\not{u}rl\,\underline{\beta}$$

$$\nabla\!\!\!\!/_4 \underline{\beta} + tr\chi\underline{\beta} + \hat{\omega}\underline{\beta} = \mathcal{D}\!\!\!\!/_1^\star \left(\rho^{(1)}, \sigma\right) + 3\eta\rho_0$$

$$\nabla\!\!\!\!/_3 \underline{\beta} + 2tr\underline{\chi}\underline{\beta} - \underline{\hat{\omega}}\underline{\beta} = -d\not{i}v\,\underline{\alpha}$$

$$\nabla\!\!\!\!/_4 \underline{\alpha} + \frac{1}{2} tr\chi\underline{\alpha} + 2\hat{\omega}\underline{\alpha} = 2\mathcal{D}\!\!\!\!/_2^\star \underline{\beta} - 3\underline{\hat{\chi}}\rho_0$$

Theorem (second version).
Solutions to the above system of gravitational perturbations decay polynomially in time to a linearized Kerr solution.

Remarks:

- decay sufficient for non-linear applications (more later)
- "stationary modes" can be computed explicitly (more later)

What was known? I

The components α and $\underline{\alpha}$ decouple and satisfy a wave equation (Teukolsky equation; $\Delta = r^2 - 2Mr$; $s = 2$)

$$\partial_r\left(\Delta\partial_r\alpha\right) - \frac{1}{\Delta}\left(r^2\partial_t - (r-M)\,s\right)^2\alpha - 4sr\partial_t\alpha$$
$$+\partial_{\cos\theta}\left(\sin^2\theta\partial_{\cos\theta}\alpha\right) + \frac{1}{\sin^2\theta}\left(\partial_\phi + is\cos\theta\right)^2\alpha = 0$$

No energy estimate known!

How to control the remaining quantities?

Remark: This decoupling remains true in Kerr.

What was known? II

On the other hand, it is known that certain metric components satisfy the Zerilli equation in frequency space

$$\frac{d^2\phi}{dr_\star^2}+\left[\omega^2-\frac{2n^2\,(n+1)\,r^3+6n^2Mr^2+18nM^2r+18M^3}{r^3(nr+3M)^2}\,(1-\mu)\right]\phi=0$$

Here ($\phi=\phi_{ml}$)

$$\frac{d}{dr_\star}=\left(1-\frac{2M}{r}\right)\frac{d}{dr}\quad\text{and}\quad n=\frac{1}{2}(l-1)(l+2)$$

One can obtain some control over ϕ but it is hard to see what one actually controls in physical space and how to go from this to the remaining metric components and their derivatives.

In summary, either

- equation for geometric quantity which decouples but neither useful estimates available nor clear how to control *everything*

OR

- good equation for artificial/ non-geometric quantity from which it is not clear how to control the other quantities.

In any case, none of these approaches leads to a uniform boundedness or decay statement for solutions to the linearized equations.

A third approach that has been tried is to derive a wave equation for the middle components ρ or $\rho^{(1)}$. Indeed ρ itself satisfies a decoupled wave equation

$$\Box_{RW}\rho = \text{quadratic terms} \equiv 0$$

But ρ does not decay and writing $\rho - \left(-\frac{2M}{r^3}\right)$ will lead to coupling with the connection coefficients and destroy the decoupling!

Remark: In the Maxwell case, this works:
See [Blue, Blue-Soffer, Andersson-Blue].

The solution will be

1. to introduce a quantity which lives **purely in physical space**
2. the quantity satisfies a “good” equation without any special gauge conditions. The analysis does not need separation of variables.
3. the quantity naturally captures the linearized Kerr modes
4. the quantity indeed eventually controls all curvature components and Ricci-coefficients which decay.

Here is the quantity:

$$\boxed{P = \not{D}_2^\star \not{D}_1^\star \left(-\rho^{(1)}, \sigma\right) + \frac{3}{4}\rho_0 tr\chi\left(\hat{\chi} - \underline{\hat{\chi}}\right)} \tag{9}$$

Here

$$\begin{aligned} \not{D}_1^\star \left(-\rho^{(1)}, \sigma\right) &= \not\nabla_A \rho^{(1)} + \epsilon_{AB} \not\nabla^B \sigma \\ \left(\not{D}_2^\star \xi\right)_{AB} &= \not\nabla_A \xi_B + \not\nabla_B \xi_A - \not{g}_{AB}\left(\not\nabla^C \xi_C\right) \end{aligned} \tag{10}$$

Observations

- P is a symmetric-traceless tensor (no $\ell = 0$ and $\ell = 1$ modes). It combines two derivatives of curvature and connection coefficients
- The linearized Kerr fields have $\sigma \neq 0$ but sit in the kernel of $\not{D}_2^\star \not{D}_1^\star$

$$\boxed{P = \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star \left(-\rho^{(1)}, \sigma\right) + \frac{3}{4}\rho_0 tr\chi \left(\hat{\chi} - \underline{\hat{\chi}}\right)}$$

We prove:

1. P decouples and satisfies a Regge-Wheeler equation for which one can prove both boundedness and integrated decay
2. P eventually controls all other quantities

Remark

We were lead to this quantity from the appendix of Chandrasekhar's 1975 paper "On the equations governing the perturbations of the Schwarzschild black hole." There, he discusses transformations (in frequency space!) that map solutions of Teukolsky to solutions of Zerilli and (in the appendix!) to Regge-Wheeler.

Once you translate back to physical space you have

$$P \sim \nabla\!\!\!/_3 \left(2T\alpha r\Omega^2\right) + \frac{2}{r}\left(1 - \frac{3M}{r}\right)\nabla\!\!\!/_3 \left(\alpha r\Omega^2\right) + 2\Omega\left(-\frac{1}{2}\Delta\!\!\!/ + K\right)\left(\alpha\Omega^2 r\right) - 3\rho_0\Omega\left(\alpha\Omega^2 r\right)$$

for

$$T = \frac{1}{2}\Omega\left(\nabla\!\!\!/_3 + \nabla\!\!\!/_4\right) \text{ and } K = r^{-2},\ \Omega = \sqrt{1 - \frac{2M}{r}},\ \rho_0 = -2Mr^{-3}.$$

The Regge-Wheeler equation

We have that $\phi = r^3 P_{AB}$ satisfies

$$\frac{1}{1-\frac{2M}{r}}\partial_u\partial_v\phi - \left(\not{\Delta} - \frac{4}{r^2}\right)\phi - \frac{6M}{r^3}\phi = 0 \tag{11}$$

The positive conserved energy is then almost obvious.
An integrated decay estimate was shown by [Blue-Soffer] (also [GH]).

Finally, one can apply the results of [DafRod] "A new physical space approach to decay for the wave equation..." to go from integrated decay to polynomial decay rates for the energy.

Obtaining bounds on all other quantities

Note that it is already non-trivial that P controls anything!

The key is to define two new quantities

$$\psi = 2\not{\mathcal{D}}_2^\star \beta + 3\rho_0 \hat{\chi} \quad \text{and} \quad \underline{\psi} = 2\not{\mathcal{D}}_2^\star \underline{\beta} - 3\rho_0 \underline{\hat{\chi}}\,. \tag{12}$$

We have the propagation equations

$$\not{\nabla}_3 \left(\psi r^3 \Omega\right) = r^3 \Omega P \tag{13}$$

and

$$\not{\nabla}_3 \left(r \Omega^2 \alpha\right) = r \Omega^2 \psi \tag{14}$$

These equations can be integrated from initial data as transport equations. You have to be careful with the weights near the horizon and near infinity.

Similar relations hold for the *bared* quantities.

From α you can obtain $\hat{\chi}$ via

$$\nabla\!\!\!/_4\hat{\chi} + tr\chi\hat{\chi} - 2\hat{\omega}\hat{\chi} = \alpha$$

This equation cannot be integrated directly from data all the way to the horizon. A version of the redshift-effect via commutation is necessary.

Again, a similar arguments can be invoked for the bared quantities.

There is a hierarchy in the equations which one exploits.

Overview

$\boxed{\text{Quantity } P}$ [Regge-Wheeler; integr. decay $\rightarrow$ polynomial rates]

$\downarrow$

$\boxed{\text{Quantities } \psi \text{ and } \underline{\psi}}$ [satisfy transport with P on the right hand side]

$\downarrow$

$\boxed{\text{Quantities } \alpha \text{ and } \underline{\alpha}}$ [satisfy transport with $\psi, \underline{\psi}$ on RHS]

$\downarrow$

$\boxed{\text{Quantities } \hat{\chi} \text{ and } \underline{\hat{\chi}}}$ [satisfy transport with $\alpha, \underline{\alpha}$ on RHS]

$\downarrow$

$\boxed{\text{Quantities } \mathcal{D}_2^\star \beta \text{ and } \mathcal{D}_2^\star \underline{\beta}}$ [from revisiting ψ, $\underline{\psi}$]

$\downarrow$

...

The non-trivial Kerr modes

Note that one can only show decay for $\not{D}_2^\star \beta$. This is again clear, since $\beta_{kerr} \neq 0$ but $\not{D}_2^\star \beta_{Kerr} = 0$.

The linearized Kerr-fields can be computed explicitly from the paper of Pretorius and Israel expressing the Kerr metric in double-null coordinates. Linearizing in the angular momentum parameter a provides the explicit expressions.

The change in mass ($l = 0$ mode) is actually quadratic.

Theorem (third version)
The following bounds hold for solutions of the linearized system.

- uniform boundedness:

$$\int_{\Sigma_{t_2^\star}} |DW^{(1)}|^2 + |W^{(1)}|^2 + |D\Gamma^{(1)}|^2 + |\Gamma^{(1)}|^2$$

$$\lesssim \int_{\Sigma_{t_1^\star}} |DW^{(1)}|^2 + |W^{(1)}|^2 + |D\Gamma^{(1)}|^2 + |\Gamma^{(1)}|^2 + \int_{\Sigma_{t_1^\star}} |DP|^2 + |P|^2$$

- integrated decay:

$$\int_{\mathcal{M}\left(t_1^\star, t_2^\star\right)} |DW^{(1)}|^2 + |W^{(1)}|^2 + |D\Gamma^{(1)}|^2 + |\Gamma^{(1)}|^2 \lesssim \int_{\Sigma_{t_1^\star}} RHS$$

[It being implicit that $W^{(1)}$ and $\Gamma^{(1)}$ have their Kerr parts removed.]

The relation with "ultimately Schwarzschildean" spacetimes

In a previous paper, I introduced a class of spacetimes which converge at a certain polynomial rate to Schwarzschild.

There was a complicated decay hierarchy:

A spacetime is $UltS_n$ if

- (the energy of) n-derivatives of curvature is bounded
- (the energy of) $(n-1)$-derivatives of curvature decays like $1/t$
- ...

I showed: $UltS_n \implies UltS_{n+1}$.

This was a conditional result. I could not *improve* all the decay rates. What was missing was an estimate "at the lowest order".

The basic idea was

$$\nabla\!\!\!/_3\alpha + \frac{1}{2}tr\chi\alpha + 2\hat{\omega}\alpha = -2\mathcal{D}\!\!\!/_2^\star\beta + 3\rho_0\hat{\chi} \tag{15}$$

$$\nabla\!\!\!/_3 T\alpha + \frac{1}{2}tr\chi T\alpha + 2\hat{\omega}T\alpha = -2\mathcal{D}\!\!\!/_2^\star T\beta + 3\rho_0 T\hat{\chi} \tag{16}$$

$$\nabla\!\!\!/_3 T\alpha + \frac{1}{2}tr\chi T\alpha + 2\hat{\omega}T\alpha = -2\mathcal{D}\!\!\!/_2^\star T\beta + 3\rho_0\left(\mathcal{D}\!\!\!/_2^\star\eta + \alpha\right) + \text{l.o.t.} \tag{17}$$

Multiplying by $T\alpha$ requires to control

$$\int\left(\mathcal{D}\!\!\!/_2^\star\eta + \alpha\right)T\alpha = \int\frac{1}{2}T\left(|\alpha|^2\right) + T\eta\cdot d\!\!/iv\,\alpha = \int\frac{1}{2}T\left(|\alpha|^2\right) + T\eta\cdot\nabla\!\!\!/_4\beta$$

Recall the boundedness statement:

$$\int_{\Sigma_{t_2^\star}} |DW^{(1)}|^2 + |W^{(1)}|^2 + |D\Gamma^{(1)}|^2 + |\Gamma^{(1)}|^2 + \int_{\Sigma_{t_2^\star}} |DP|^2 + |P|^2$$
$$\lesssim \int_{\Sigma_{t_1^\star}} |DW^{(1)}|^2 + |W^{(1)}|^2 + |D\Gamma^{(1)}|^2 + |\Gamma^{(1)}|^2 + \int_{\Sigma_{t_1^\star}} |DP|^2 + |P|^2$$

The above ultimately Schwarzschildean technique can be used to obtain a boundedness statement for the linearized fields without any derivative loss. This is important for non-linear applications.

Nonlinear applications

Given the present result as well as the ultimately Schwarzschildean paper (where the non-linearities are understood) we can address

Open Problem: Consider axisymmetric initial data for the vacuum Einstein equations sufficiently close to Schwarzschildean data and such that the angular momentum vanishes.
Prove that the maximum development contains a black hole which dynamically converges to a member of the Schwarzschild spacetime.

Remaining Difficulty: How to determine the final mass.

Further directions

- the Kerr case: analogue of P?
- ultimately Kerr (work in progress)