Stefanos Aretakis

Fig 1

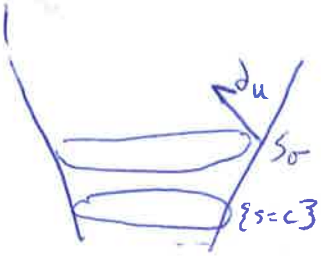


Fig 2.

Fig 3

Fig 4

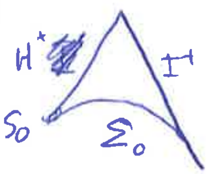


Fig 5

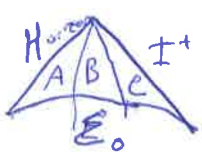


Fig 6

Fig 7

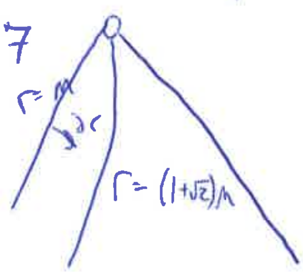


Fig 8

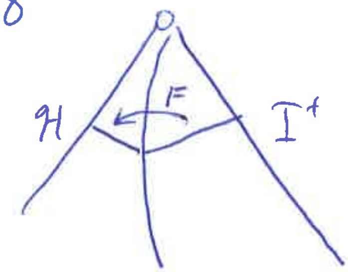


Fig 9

Fig 9

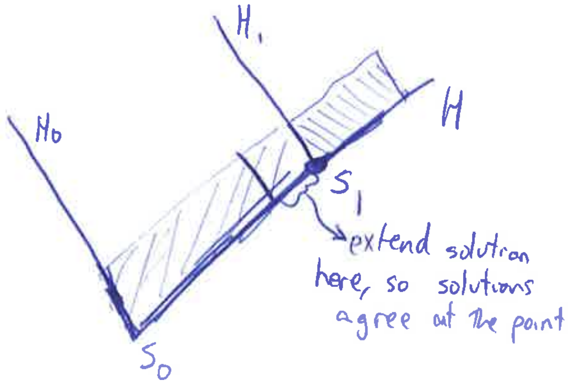
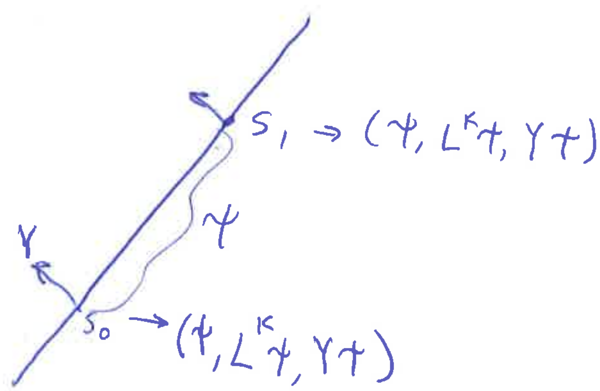


Fig 10



CONSERVATION LAWS FOR THE WAVE EQUATION ON NULL HYPERSURFACES AND APPLICATIONS

STEFANOS ARETAKIS

These conservations laws are related to conserved charges in Maxwell's equations. Those arise from closed 2-form.

These conservation laws are the obstruction to gluing of the null constraint equations.

Outline:

(1) Examples
(2) Applications to black holes
(3) General theory and gluing problem

Purposes: We are studying the stability of the wave equation on black holes, and also the general theory.

Examples: 1. $\mathbb{R}^{3+1}$. Take a solution to $\Box\psi = 0$ in coordinates $(u = t - r, v = t + r, \theta)$. Then $\int_{S_v} \frac{1}{r^2}\partial_u(r\psi)dV$ is conserved independent of v. Fig 1. [Warning: his u's and v's were nearly identical throughout, so I may accidentally interchange them.]

2. "Newman-Pensrose" constant: Take (u, r, θ). Then $\forall$ solutions to $\Box\psi = 0$,

$$\lim_{r\to\infty} \int_{S_u} r^2\partial_r(r\psi)\sin\theta d\theta$$

is preserved. See fig 2. We can write

$$\psi(u, r, \theta) = \alpha_2(u, \theta)/r + \alpha_2(u, \theta)/r^2 + O(1/r^3).$$

Here α_1 is known as a radiation field. The second term is what gives the conserved quantity, i.e. the conserved quantity is $\int_{S^2} \alpha_2(u, \theta)\sin\theta d\theta$.

3. Kerr family: a, m are the parameters. $|a| = m$ is extremal case. Take (v, r, θ) for coordinates. We have $r_{H^+} = m$. For any ψ such that $\Box\psi = 0$, then

$$H_0[\psi] := \int_{S_v} \left(\partial_r\psi + \frac{\sin^2\theta}{4}(\partial_v\psi) + \frac{1}{2m}\psi\right) dV$$

is independent of v. See fig 3.

Fig 4. We want to find quantities on S_0 (in figure 4) since then never decrease? [I'm not sure I got this right.]

Applications:

Evolution of linear waves on black holes. We split our spacetime into 3 regions; the area near the horizon H, called A, B in the middle and the region C near infinity. (see figure 5)

Difficulties:

(1) Redshift effect in A
(2) Trapping effect in B
(3) Understanding AF region of C
(4) superradiance
(5) low-frequency obstructions

These are here for all of the Kerr metrics, but in the extremal case these difficulties couple. Others have shown that you get decay for non-extremal case. In particular, $|\psi| < \frac{1}{2}$ and $|D^k\psi| < \frac{1}{2}$ for $\{r \geq r_H\}$.

In the extremal case, 1. there is no redshift effect i.e. the surface gravity k becomes zero. 2. The trapping reaches H, so A does not exist. This is because there exists a sequence γ_{r_0}, null geodesics, that are supported on $\{r = r_0\}$, with $r_0 \to r_H$. Sbierski has constructed approximate solutions to the wave equation, finite regions that stay close to these null geodesics. See fig 6. There exists ψ_λ with $E[\psi\lambda] \sim c$.

3. Trapping and superraddiance are coupled. This is because geodesics approaching the horizon become superradiant.

Let us look at symmetric solutions to $\Box\psi = 0$ with $\Phi\psi = 0$. Thus $\gamma_r \perp \Phi := \partial_\phi$. See fig 7. In this case, there is no superradiance. We have $|\psi| < 1/2^{\frac{3}{5}}$ for sufficiently regular data. Also, $|\partial_v\psi| < 1/2^{\frac{3}{5}}$. This immediately gives us that $\sup_{S_v}(\partial_r\psi) \geq cH_0$, looking at a conserved quantity on Kerr.

We can do better if we look at higher derivatives. $\sup_{S_v}|\partial_r\partial_r\psi| \geq cH_0 v$. The more derivatives, the better growth. Thus we have an instability, since this implies blowup.

Recently, Bizon-Friedrich, Reall, et al have shown that there is a correspondence in the following sense: see fig 8. For an extremal Kerr there exists a diffeomorphism $F : D \to D$ sending the pieces to each other as drawn. For any solution of $\Box\psi = 0$ with $\Phi\psi = 0$, then $\Box(F^*(\phi)) = 0$. The conserved quantity on the I^+ thus corresponds to the one on the horizon.

Reall-Lucietti have derived gravitational conservation laws for extremal Kerr.

What happens if we consider initial data which is compactly supported? Then the conserved quantity could be 0, so a priori the derivatives of ψ could decay. But it turns out that even if $H_0[\psi] = 0$, then $\sup_{S_v}|\partial_v\partial_v\partial_v\psi| \to \infty$ for "generic" data.

We now consider (M, g), not necessarily satisfying Einstein equations. See fig 9. Take a foliation of H, a null hypersurface, which is a collection $S = (S_v)_v$, and can be written as $\langle S_0, \Omega, L_{geod}\rangle$, where L_{geod} satisfies the geodesic equation. Why? We can write $L = \Omega^2 L_{geod}$. If $\{v = 0\} = S_0$, then $S_v = \{v = v_0\}$.

The induced metric $\not{g}$. We have the metric induced by the diffeomorphism from S^2 to S_0. This metric is $\not{g}_{S^2(1)}$. Let Y^s be the null vector normal to to S_v, normalized such that $g(Y^s, L_{geod}) = -1$.

Conservation laws:

We define

$$W^S := \{\Theta \in C^\infty(H) : L\Theta = 0, \forall \psi \text{ s.t. } \Box\psi = 0, \partial_v \left(\int_{S_v} Y^s(\phi\psi)\Theta d\slashed{g}_{S^2(1)} \right) = 0\}$$

where there exists a unique $\hat{\slashed{g}}$ with $d\hat{\slashed{g}} = d\slashed{g}_{S^2(1)}$, with $\slashed{g} = \phi^2 \hat{\slashed{g}}$.

1. The only conservation laws with respect to S that H can admit are of the above form, i.e. $W^S \neq 0$.

2. We can characterize null hypersurfaces that have conservation laws. Given a foliation S of H, we define $O^S : C^\infty(\psi) \to C^\infty(\psi)$ by

$$O^S\psi := \Omega^2 \slashed{\Delta}\psi + (\slashed{\nabla}\Omega^2 + 2\Omega^2 J^\#)\slashed{\nabla}\psi + \left[2\slashed{\text{div}}(\Omega J^\#) + \partial_v(\Omega \text{tr}\underline{\chi}) + \frac{1}{2}(\Omega\text{tr}\chi)(\Omega\text{tr}\underline{\chi}) \right] \psi$$

where J is the torsion of S_v and $\text{tr}\chi$, $\text{tr}\underline{\chi}$ are the null mean curvatures.

Let

$$U^S := \{\Theta \in C^\infty(H) : L\Theta = 0, O^S_v \left(\frac{1}{\phi}\Theta \right) = 0 \, \forall v\}$$

where $O^S_v = O^S|_{S_v}$.

Then $W^S = U^S$, and so we get a conservation law.

3. What happens if we refoliate? Assume that, with respect to S, we have a conservation law. If we refoliate, S', do we still have a conservation law? If S, S' are two foliations, then $O^S \left(\frac{1}{\phi}\Theta \right) = O^{S'} \left(\frac{1}{\phi}\Theta \right)$ for any Θ such that $L\Theta = 0$.

We then sweep an element of S' with foliations from the other. Take $\Theta \in W^S$. We apply property 3, and get $O^{S'} \left(\frac{1}{\phi}\Theta \right) = 0$, and so we know something about the kernel of the operator. Thus $W^S = W^{S'}$. We thus get a conservation law for any foliation, and also the conserved quantity is equal.

4. The conservation laws are the only obstruction to the characteristic gluing problem. If we give data on shaded lines (fig 9), we get solution in shaded rectangle. If we want to glue solutions together, this is the problem. Can only glue solutions (see fig 10) if the conserved quantities are conserved. This is the only obstruction.