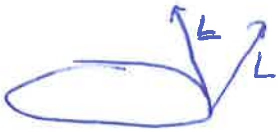
Klainerman

Fig 1



$\left.\begin{array}{l}\chi \\ \underline{\chi}\end{array}\right\} \to$ null 2nd fund forms.

$\hat{\chi}, \hat{\underline{\chi}}$ are traceless parts.

Fig 2

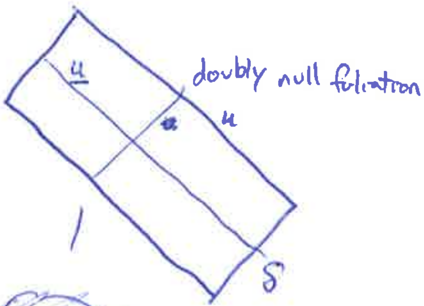
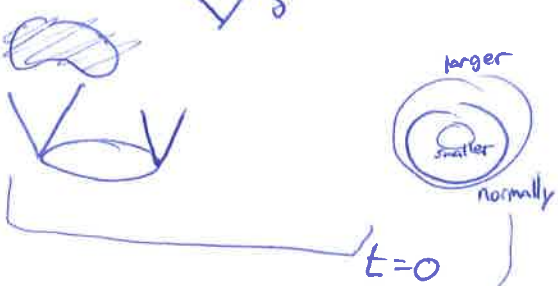


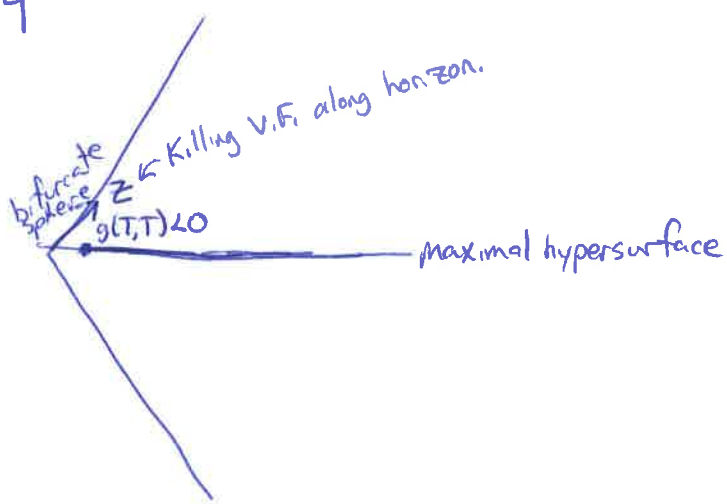
Fig 3



but if deformed

the outgoing here will have to shrink locally.

Fig 4



ON THE FORMATION OF TRAPPED SURFACES

SERGIU KLAINERMAN

Initial data slide: We can specify the traceless part of the null 2nd fundamental forms, as in figure 1.

Main ideas slide: See figure 2.

Intuition of result (not slide): Figure 3. If I start with something deformed, the area is going down in a local region, and so $\mathrm{tr}\chi$ is negative locally. But this won't work globally since can't do it at all points. This result says I can deform the initial surface everywhere except around a point enough so that I get $\mathrm{tr}\chi$ negative everywhere except the point, and then use the short pulse to get the rest of the trapped surface. $\mathrm{tr}\underline{\chi}$ is controlled from the first part, the existence result.

Local rigidity slide: Need to extend Z. If I can extend to region where T is timelike, I can appeal to analyticity, or the result on the slide.

For 1st result, we want the space to be strongly gravitating, but we want double null foliation to exist. Is it difficult to avoid caustics, but still strongly gravitating enough to make trapped surfaces? No. The existence part controls this, and it is hard, but past that, it is not difficult.

ON THE FORMATION OF TRAPPED SURFACES

Sergiu Klainerman

Princeton University

November 19, 2013

BLACK HOLES REALITY TESTS.

a. RIGIDITY. Does the Kerr family $\mathcal{K}(a, m)$, $0 \le a \le m$, exhaust all possible, **stationary**, asymptotically flat, vacuum black holes ?

b. STABILITY. Is the Kerr family stable under arbitrary small perturbations ?

c. COLLAPSE. Can black holes form starting from reasonable initial data configurations ? Formation of trapped surfaces.

BLACK HOLES REALITY TESTS

RIGIDITY.

STABILITY.

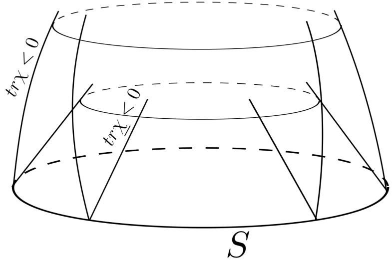
COLLAPSE. Can black holes form starting from reasonable initial data configurations ? Formation of trapped surfaces.

- Penrose Singularity Theorem(1969)
- Christodoulou's Trapping Theorem(2008)
- Kl-Rodnianski(2010)
- Kl-Luk-Rodnianski(2013)

PENROSE SINGULARITY THEOREM

THEOREM. Space-time (M, g) cannot be future null geodesicaly complete, if

- $\mathrm{Ric}(g)(L, L) \geq 0, \qquad \forall L$ *null*
- M contains a non-compact Cauchy hypersurface
- M contains a closed **trapped** surface S



Null second fundamental forms $\chi, \underline{\chi}$

QUESTIONS

- Quantitative version of the incompleteness theorem ?
- Can trapped surfaces form in evolution ? In vacuum ?
  Does the existence of a trapped surface implies the presence of a Black Hole ?

  True if **weak cosmic censorship** holds true.
- *Significance of the uniformity condition ?*

  Can singularities form starting with **non-isotropic**, initial configurations?

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FORMATION OF TRAPPED SURFACES

PROBLEM. Specify an open set of regular initial conditions, free of trapped surfaces, on a space-like or **null** configuration whose future development contains a trapped surface.

DIFFICULTIES

- **Heuristics ?** In the absence of spherical symmetry it is not at all clear how such a large, **uniform** distortion can be produced.
- **Semi-Global.** Need to control the MFGHD of an initial data for a **far longer** time than that provided by the classical existence results [Y. C. Bruhat(1952)), Rendall(1990)]

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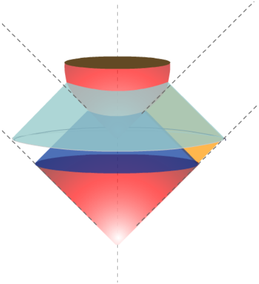
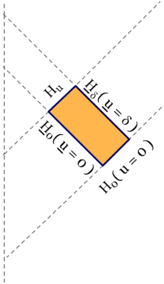
MAIN RESULTS

THEOREM[[Christ(2008)]. ($\exists$) open set of regular, vacuum, data whose MGFHD contains a trapped surface.





1. Specify **short pulse** characteristic data, for which one can prove a general semi-global result, with **detailed control**.
2. If, **in addition**, the data is sufficiently large, **uniformly** along all its null geodesic generators, a trapped surface must form.
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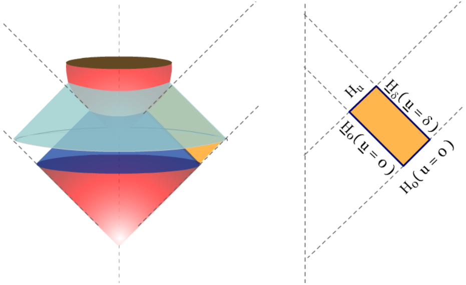

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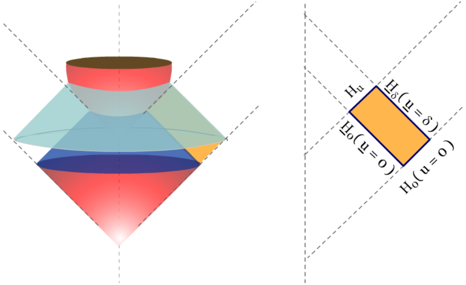

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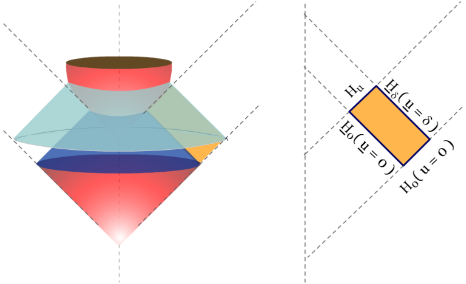

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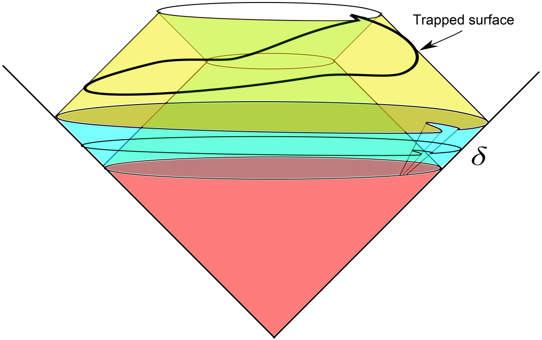

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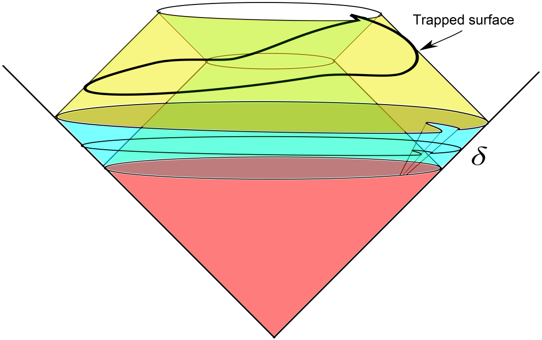

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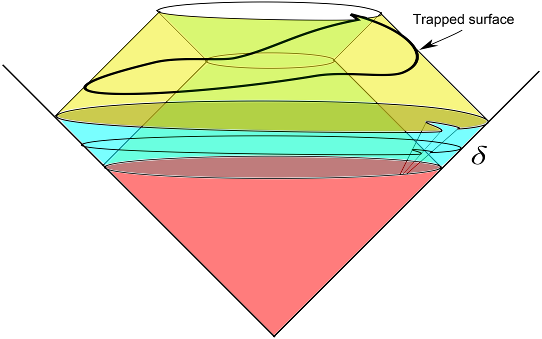
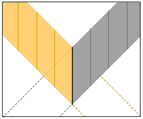


CHARACTERISTIC INITIAL VALUE PROBLEM

Characteristic Data. $\mathrm{Ric}(g) = 0.$

- Hypersurfaces $H_0 \cup \underline{H}_0 \subset \mathbb{R}^{1+3}$, $S_{0,0} = H_0 \cap \underline{H}_0$.



- Regular foliations $S_u \subset \underline{H}_0$, $S_{\underline{u}} \subset H_0$
- Conformal metrics $([\gamma]_u, S_u)$, $([\gamma]_{\underline{u}}, S_{\underline{u}})$.

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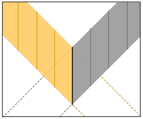

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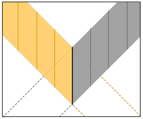

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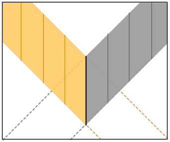
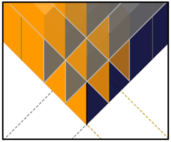
CHARACTERISTIC INITIAL VALUE PROBLEM

THEOREM[Rendall]. For any given, regular, characteristic initial data set there exists a unique, **future** development near $S_{0,0}$, foliated by a double null foliation $(u, \underline{u})$.

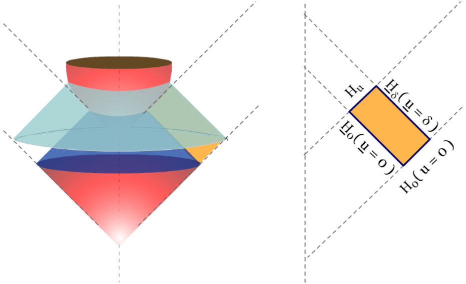




1. Max. future development $\mathcal{D}(u_*, \underline{u}_*)$, $0 \le u \le u_*, 0 \le \underline{u} \le \underline{u}_*$.
2. Instead of $[\gamma]_u$, $[\gamma]_{\underline{u}}$ one can prescribe the shears $\hat{\underline{\chi}}_0$, $\hat{\chi}_0$.

FORMATION OF TRAPPED SURFACES

THEOREM[Chr. 2008]. Specify δ-**regular** initial data on $H_0 \cup \underline{H}_0$ whose future development $\mathcal{D}(u_*, \delta)$ contains a trapped surface.



REMARKS.

1. Data trivial on $\underline{H}_0$, **short pulse** on $H_0 = \{u = 0,\ 0 \le \underline{u} \le \delta\}$, uniformly distributed in all directions.
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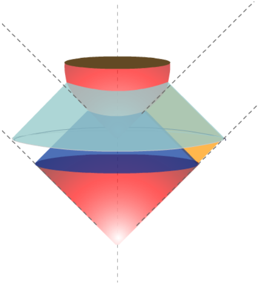

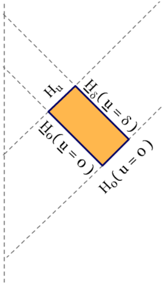

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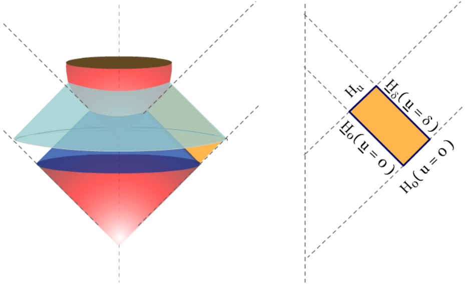

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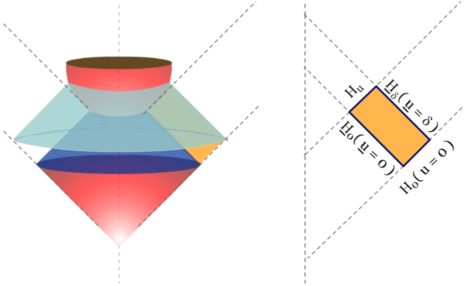

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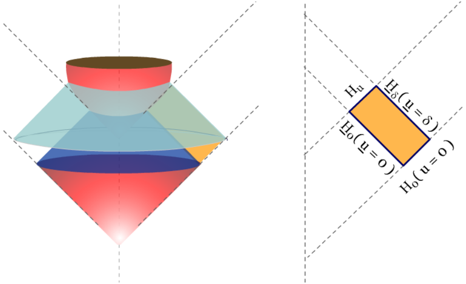

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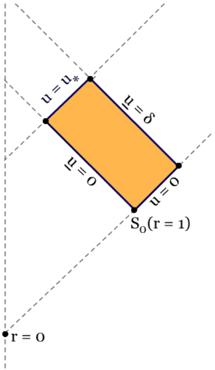
INITIAL DATA

- Flat on $\underline{H}_0(\underline{u} = 0)$.
- δ- pulse on $H_0(u = 0)$.



Christodoulou's δ-pulse. $\hat{\chi}_0(\underline{u}, \omega) = \delta^{-1/2} h(\underline{u}/\delta, \omega)$.

INITIAL DATA

Definition[Christodoulou's $C(\delta, B)$-data]

$$\sum_{i\leq 5}\sum_{k\leq 3}\delta^{\frac{1}{2}+k}||\nabla_{\underline{u}}^{k}\nabla^{i}\hat{\chi}_0||_{L^\infty(H_0)}\leq B.$$

Definition[Kl-Rodn's $KR(\delta, B)$-data]

$$\begin{aligned}\sum_{0\leq k\leq 2}\delta^{1/2}\|(\delta\nabla_4)^k\hat{\chi}_0\|_{L^2(H_0)} &\leq B\\ \sum_{0\leq k\leq 1}\sum_{1\leq m\leq 4}\delta^{1/2}\|(\delta^{1/2}\nabla)^{m-1}(\delta\nabla_4)^k\,\nabla\hat{\chi}_0\|_{L^2(H_0)} &\leq B\\ \delta^{1/2}\|\hat{\chi}_0\|_{L^\infty(H_0)} &\leq B\end{aligned}$$

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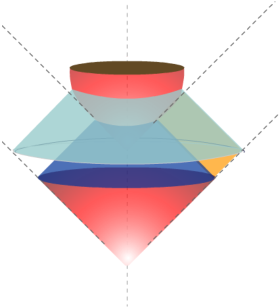
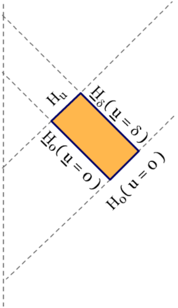
ISOTROPIC TRAPPING THEOREM

THEOREM[Christ(2008), Kl-Rodn(2010)]

1. Given $B > 0$, $u_* < 1$ there exists $\delta > 0$ sufficiently small such that if the data verify $C(\delta, B)$ or $KR(\delta, B) \Rightarrow$ the maximal future development contains $\mathcal{D}(u_*, \delta)$.



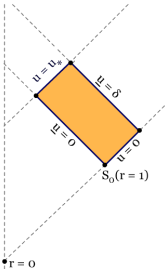


ISOTROPIC TRAPPING THEOREM

1. Given $B > 0$, $u_* < 1$, $\exists \delta > 0$ small s.t. if data verify $C(\delta, B)$ or $KR(\delta, B) \Rightarrow$ the maximal future development $\supset \mathcal{D}(u_*, \delta)$.
2. If in addition, $M_0(\omega) := \int_0^\delta |\hat{\chi}_0|^2(\underline{u}', \omega) d\underline{u}'$, verifies,

$$\inf_{\omega \in S_{0,0}} M_0(\omega) \geq M_* > 0,$$

   Then, for any $0 < u_* < 1$, $\exists \delta = \delta(B, M_*) > 0$ such that surface $S_{u_*,\delta} \subset \mathcal{D}(u_*, \delta)$ is trapped.
3. A companion result can be proved for formation of scars.



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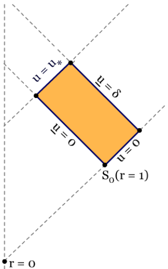

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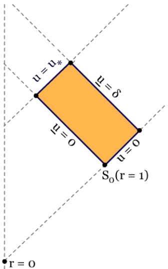

ISOTROPIC TRAPPING THEOREM

1. Given $B > 0$, $u_* < 1$, $\exists \delta > 0$ small s.t. if data verify $C(\delta, B)$ or $KR(\delta, B) \Rightarrow$ the maximal future development $\supset \mathcal{D}(u_*, \delta)$.
2. If in addition, $M_0(\omega) := \int_0^\delta |\hat{\chi}_0|^2(\underline{u}', \omega) d\underline{u}'$, verifies,
$$\inf_{\omega \in S_{0,0}} M_0(\omega) \geq M_* > 0,$$
Then, for any $0 < u_* < 1$, $\exists \delta = \delta(B, M_*) > 0$ such that surface $S_{u_*, \delta} \subset \mathcal{D}(u_*, \delta)$ is trapped.
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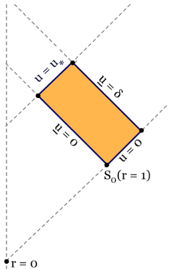

MAIN IDEAS

1. Based on a continuity argument, one derives detailed information of all connection coefficients and curvature components with respect to an adapted null frame.
2. Transport equations

$$
\begin{aligned}
\frac{d}{d\underline{u}}\mathrm{tr}\,\chi &= -|\hat{\chi}|^2 + \mathrm{err}(\delta) \\
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\end{aligned}
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Show,

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\mathrm{tr}\,\chi(u,\underline{u},\omega) = \frac{2}{r(u,0)} - \frac{1}{r^2(u,0)}\int_0^{\underline{u}} |\hat{\chi}_0|^2(\underline{u}',\omega)\,d\underline{u}' + O(\delta)
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Need $\boxed{\frac{2(r_0-u)}{r_0^2} < \inf_{\omega\in\mathbb{S}^2} M_0(\omega)} < \frac{2}{r_0}$

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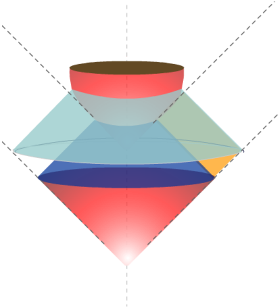
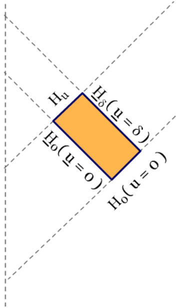
UN-ISOTROPIC TRAPPING

Theorem[Kl-Luk-Rodn(2013)] Assume $C(\delta, B)$. If in addition,

$$\sup_{\omega \in S_{0,0}} M_0(\omega) \geq M_* > 0.$$

then, for any, $0 < u_* < 1$, $1 - u_*$ small, $\exists \delta = \delta(B, M_*) > 0$ such that $\mathcal{D}(u_*, \delta)$ contains a trapped surface.





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- Uses all ingredients in Christodoulou's theorem combined with a new deformation argument along $\underline{H}_\delta$.
- Surface $\underline{u} = \delta, r = R(\omega)$ $[\ R \in \mathcal{C}^2(S_0)\ ]$ is trapped if

$$\begin{aligned} 1 - u_* &< R < 1 \\ -\Delta R + R^{-1}|\nabla R|^2 + R &< 2^{-1} M_0 \end{aligned}$$

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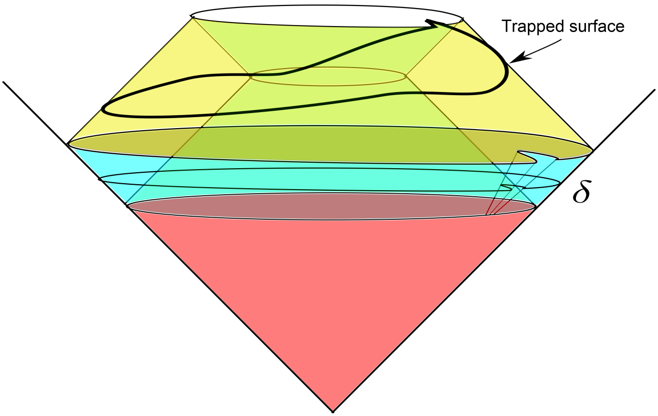
DEFORMED TRAPPED SURFACE



REALITY TESTS

RIGIDITY.
Does the Kerr family $\mathcal{K}(a, m)$, $0 \leq a \leq m$, exhaust all possible, **stationary**, asymptotically flat, vacuum black holes ?

STABILITY.

COLLAPSE.

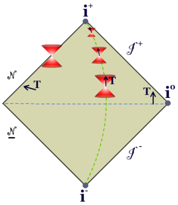
STATIONARY BLACK HOLES

Stationary, asymptotically flat, solutions of the EVE,

$$\mathbf{Ric}(g) = 0.$$

DEFINITION [External Black Hole]

- Asymptoticaly flat, globally hyperbolic, Lorentzian manifold with boundary (M, g), diffeomorphic to the complement of a cylinder $\subset \mathbb{R}^{1+3}$.



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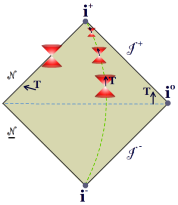

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KERR FAMILY $\mathcal{K}(a, m)$

Boyer-Lindquist (t, r, θ, φ) coordinates.

$$-\frac{\rho^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\left(d\varphi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2,$$

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

Stationary. $\mathbf{T} = \partial_t$
Axisymmetric. $\mathbf{Z} = \partial_\varphi$

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RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m)$, $0 \leq a \leq m$, exhaust all **stationary**, asymptotically flat, **regular** vacuum black holes.

- True in the static case. [Israel, Bunting-Masood ul Ulam]
- True in the axially symmetric case [Carter-Robinson]
- True in general, under an **analyticity** assumption [Hawking]
- True close to a Kerr space-time [Alexakis-Ionescu-Kl]

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- Construction of a second symmetry.
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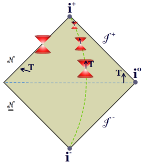
NEW RIGIDITY RESULT

THEOREM. Assume (M, g) nondegenerate, regular, black hole with a stationary Killing v-field $\mathbf{T}$ and $\|\mathbf{g}(\mathbf{T}, \mathbf{T})\|_{L^\infty(\mathbf{S}_0)}$ small.

$\Rightarrow (M, g)$ is stationary, axially symmetric, hence a $\mathcal{K}(a, m)$ with small a.

MAIN STEPS

1. (M, g) admits a a second rotational Killing vector-field $\mathbf{Z}$ which commutes with $\mathbf{T}$, in a small neighborhood Ω of the horizon.
2. If $\|\mathbf{g}(\mathbf{T}, \mathbf{T})\|_{L^\infty(\mathbf{S}_0)}$ sufficiently small $\Rightarrow \mathbf{T}$ is strictly time-like in the complement of Ω.



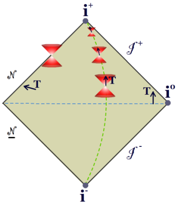
T BECOMES TIMELIKE !

Assume. Maximal hypersurface Σ_0, passing through bifurcate sphere S_0. Decompose $\mathbf{T}$,

$$\mathbf{T} = nT_0 + X, \qquad \mathbf{g}(\mathbf{T}, \mathbf{T}) = -n^2 + |X|^2$$

$$\begin{aligned} \nabla_i X_j + \nabla_j X_i &= 2nk_{ij}. \\ \Delta n &= |k|^2 n, \end{aligned}$$

Remark. $n|_{S_0} = 0, \quad n = 1$ at infinity.



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$$\begin{aligned}\nabla_i X_j + \nabla_j X_i &= 2nk_{ij}.\\ \Delta n &= |k|^2 n,\end{aligned}$$

Step 1. By Hopf Lemma $\nu(n) > 0$ in a quantitative fashion.
Step 2. $\int_{\Sigma_0} n|k|^2$ is small.
Integrate the identity $n|k|^2 = k^{ij} n k_{ij} = \nabla^i X^j k_{ij} = \nabla^i (X^j k_{ij})$.

Step 3. Use Steps 1,2 to show that k is uniformly small. **Step 4.**

Show, by a propagation argument, that X remains small. **Step 5.**

Deduce that $\mathbf{T}$ becomes strictly time-like away from a small neighborhood of S_0.

LOCAL RIGIDITY

Theorem[Ionescu-K(2011)]

$(\mathbf{M}, \mathbf{g})$ **Ricci flat**, pseudo-riemannian manifold; (O, Z) verify:

- **A1** Z Killing v-field in O,
- **A2** ∂O is **strongly pseudo-convex** at $p \in \partial O$

$\Rightarrow Z$ extends as a Killing vector-field to a neighborhood of p.

Pseudo-convexity

$O \subset \mathbf{M}$ is strongly pseudo-convex at $p \in \partial O$ if it admits defining function f at p, s.t. for any $X \neq 0 \in T_p(\mathbf{M})$, $X(f)(p) = 0$ and $\mathbf{g}(X, X) = 0$, we have

$$\mathbf{D}^2 f(X, X)(p) < 0.$$

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