Fig 1

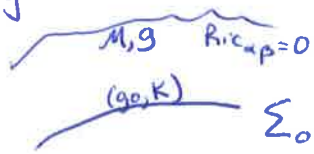


Fig 2

Fig 2

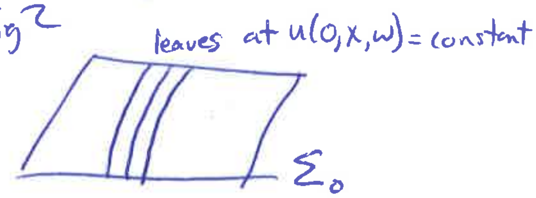


Fig 3

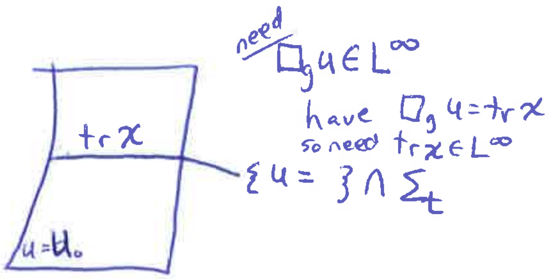
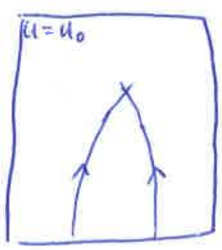


Fig 4



THE RESOLUTION OF THE BOUNDED L^2 CURVATURE CONJECTURE IN GENERAL RELATIVITY

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Cauchy Problem for EE slide: Dimension of M is 4 and g is $(-+++)$.

Ex: $(\mathbb{R}^{1+3}, m)$ where $m = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$.

See figure 1.

The bounded L^2 curvature theorem slide: The $H^{2+\epsilon}$ regularity is optimal for general quasilinear wave equations. Special properties of the Einstein equations are used to get the H^2 for this result.

"Invariant" here means coordinate invariant.

Strategy of the proof slide: We want $\Box_g \phi = Q(\phi, \phi)$, with Q to have a specific structure, called null structure. For instance, $\Box\phi = (\phi_t)^2$ does not have null structure. However, $Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_i \psi \partial_j \phi$ does have this structure.

For step B: If I want to control this, I could use something like

$$\|\partial\phi\partial\phi\|_{L^2(M)} \leq \|\partial\phi\|_{L^\infty_t L^2(\Sigma_t)} \|\partial\phi\|_{L^\infty_t L^2(\Sigma_t)},$$

but this does not exploit the structure of the quadratic form, so it must be the wrong one to use here. (Here Σ_+ is the foliation.)

Building a parametrix slide: We have $\Box_g \phi = 0$ and $\phi_{\Sigma_0} = \phi_0$ and $T\phi|_{\Sigma_0} = \phi_1$.

Step C: construction and control of the parametrix slide: In the flat case: $u_\pm = \pm t + x.\omega$ and

$$f_\pm(\lambda\omega) = \frac{1}{2}\left(\hat{\phi}_-(\lambda\omega) \mp \frac{i}{\lambda}\hat{\phi}_1(\lambda\omega)\right).$$

Next slide: If we differentiate the eikonal equation twice, we get a curvature term. If we do, we get it in L^2. Thus we can't do more derivatives. But what about ω? We only get a limited number of derivatives with respect to ω, unfortunately, otherwise it'd be much easier.

Third bullet on that slide: We will prescribe the leaves of Σ_0 as in figure 2. Also see figure 3, and we need to prescribe that $\mathrm{tr}\chi \in L^\infty$ along this foliation. If we foliate by solutions of minimal surface equation, it doesn't quite work, since it doesn't know about normal directions. But something related to mean curvature flow works to get this.

Fourth: Why do we need the lower bound on injectivity radius? The method of characteristics fails where null geodesics cross as in figure 4, which this lower bound controls.

In other dimensions, would expect you would need 1/2 derivative more.

The resolution of the bounded L^2 curvature conjecture in general relativity

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Cauchy Problem for EE

$(\mathcal{M}, \mathbf{g})$ Lorentzian, $\mathbf{R}$ curvature tensor of $\mathbf{g}$

Einstein Vacuum equations: $\mathbf{Ric}_{\alpha\beta} = 0$

Wave coordinates: $\square_{\mathbf{g}} x^\alpha = \frac{1}{\sqrt{|\mathbf{g}|}} \partial_\beta (\mathbf{g}^{\beta\gamma} \sqrt{|\mathbf{g}|} \partial_\gamma) x^\alpha = 0, \alpha = 0, 1, 2, 3$

$\square_{\mathbf{g}} \mathbf{g}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(\mathbf{g}, \partial \mathbf{g})$, $\alpha, \beta = 0, 1, 2, 3$, with $\mathcal{N}_{\alpha\beta}$ quadratic w.r.t $\partial \mathbf{g}$

Cauchy data: (Σ_0, g_0, k) where $\Sigma_0 = \{t = 0\}$, $\mathbf{g}(0, .) = g_0$, $\partial_t \mathbf{g}(0, .) = k$

Question: Under which regularity do we have local existence for EE?

The bounded L^2 curvature theorem

Theorem [KRS (2012)]: Let (Σ_0, g_0, k) *with* $R \in L^2(\Sigma_0)$ *and* $\nabla k \in L^2(\Sigma_0)$. *Then, EE are WP*

Motivations:

- First WP result for a quasilinear wave equation below $H^{2+\epsilon}$, and first to exploit the full nonlinear structure of the equation
- The assumptions $R \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$ are invariant
- Rather than a WP result, it can be viewed as a breakdown criterion. In particular, $\mathbf{R} \in L^2$ is a fundamental quantity controlling singularity formation
- There is some criticality in this problem: the control of the Eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ requires $\mathbf{R} \in L^2$

Strategy of the proof

A *Recast the EE as a quasilinear Yang-Mills theory*

B *Prove appropriate bilinear estimates for solutions to* $\Box_{\mathbf{g}}\phi = 0$

C *Construct a parametrix for* $\Box_{\mathbf{g}}\phi = 0$, *and obtain the control of the parametrix*

D *Prove a sharp* $L^4(\mathcal{M})$ *Strichartz estimate for the parametrix*

Achieve Steps B, C and D only assuming L^2 bounds on $\mathbf{R}$

This requires to exploit the full structure of the Einstein equations

Step A: EE as a quasilinear Yang-Mills theory

Let e_α an orthonormal frame on $\mathcal{M}$, i.e. $\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta}$

Let $(\mathbf{A}_\mu)_{\alpha\beta} := (\mathbf{A})_{\alpha\beta}(\partial_\mu) = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha)$

$$\mathbf{R}(e_\alpha, e_\beta, \partial_\mu, \partial_\nu) = \partial_\mu(\mathbf{A}_\nu)_{\alpha\beta} - \partial_\nu(\mathbf{A}_\mu)_{\alpha\beta} + (\mathbf{A}_\nu)_\alpha{}^\lambda(\mathbf{A}_\mu)_{\lambda\beta} - (\mathbf{A}_\mu)_\alpha{}^\lambda(\mathbf{A}_\nu)_{\lambda\beta}$$

$$\mathbf{D}^\mu \mathbf{R}_{\alpha\beta\mu\nu} = 0 \text{ (consequence of Bianchi identities + EE)}$$

$$(\square_{\mathbf{g}} \mathbf{A})_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) + [\mathbf{A}^\mu, \mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu] + \mathbf{A}^3$$

We choose the Coulomb gauge $\nabla^j A_j = 0$

We need a procedure to scalarize the tensorial wave equation and to project on divergence free vectorfields without destroying the null strucure

Step B: the bilinear and trilinear estimates

We need to control scalar functions ϕ solutions of

$$\Box_{\mathbf{g}}(\phi) = \text{null forms } + l.o.t$$

$\Rightarrow$ control the energy estimate + prove bilinear estimates

To prove these bilinear estimates in a quasilinear setting:

- write ϕ by iterating the basic parametrix of step C (construction and control of the parametrix)
- Rethink the proof of bilinear estimates in the quasilinear setting
- Prove a sharp $L^4(\mathcal{M})$ Strichartz estimate (step D)
- prove a trilinear estimate to control the energy estimate

Building a parametrix for $\Box_{\mathbf{g}}(\phi) = 0$

Let a plane wave $e^{i\lambda u(t,x,\omega)}$ with $\lambda \in [0, +\infty)$ and $\omega \in \mathbb{S}^2$ parameters corresponding to Fourier variables in $\mathbb{R}^3$ in spherical coordinates

$$\Box_{\mathbf{g}}(e^{i\lambda u}) = \Big(- \lambda^2 \mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u + i\lambda \Box_{\mathbf{g}} u \Big) e^{i\lambda u}$$

For u a solution of the Eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$, we have:

$$\Box_{\mathbf{g}}(e^{i\lambda u}) = i\lambda \Box_{\mathbf{g}} u e^{i\lambda u}$$

This yields in general an approximate solution to $\Box_{\mathbf{g}}(\phi) = 0$. We then superpose these plane waves to generate any initial data

Step C: construction and control of the parametrix

$$S(t,x)=\sum_{\pm}\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u_\pm(t,x,\omega)}f_\pm(\lambda\omega)\lambda^2 d\lambda d\omega$$

where $\mathbf{g}^{\alpha\beta}\partial_\alpha u_\pm\partial_\beta u_\pm=0$ on $\mathcal{M}$ such that $u_\pm(0,x,\omega)\sim x.\omega$ when $|x|\to+\infty$ on Σ_0

Construction: for any (ϕ_0,ϕ_1) there exists $f_\pm$ such that

$S(0,.)=\phi_0$, $TS(0,.)=\phi_1$ and $\|\lambda f_\pm\|_{L^2(\mathbb{R}^3)}\lesssim\|\nabla\phi_0\|_{L^2(\Sigma_0)}+\|\phi_1\|_{L^2(\Sigma_0)}$

$$E(t,x)=\Box_{\mathbf{g}}S(t,x)=i\sum_{\pm}\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u_\pm(t,x,\omega)}\Box_{\mathbf{g}}u_\pm(t,x,\omega)f_\pm(\lambda\omega)\lambda^3 d\lambda d\omega$$

Control of the error term: $\|E\|_{L^2(\mathcal{M})}\lesssim\|\lambda f_+\|_{L^2(\mathbb{R}^3)}+\|\lambda f_-\|_{L^2(\mathbb{R}^3)}$

Step C: construction and control of the parametrix

- *Goal: Achieve Step* **C** *only assuming* L^2 *bounds on* **R**. *This requires to exploit the full structure of Einstein equations*

- *The regularity in* ω *of* $u_\pm$ *obtained in Step* **C** *is limited*

- *A careful choice of* $u_\pm(0, x, \omega)$ *(related to the mean curvature flow) allows us to "squeeze" as much regularity in* x *and* ω *as possible*

- $\mathbf{R} \in L^2$ *is minimal to obtain a lower bound on the radius of injectivity of level surfaces of the phase* $u_\pm$

- *Step* **C** *requires* L^2 *bounds for Fourier integral operators, and in turn several integration by parts. Classical proofs (*TT^* *and* T^*T *arguments) would fail by far*