Fig 1

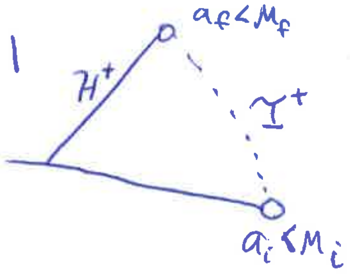


Fig 2

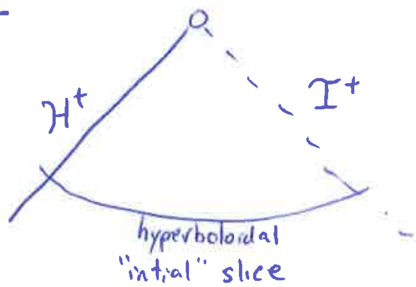
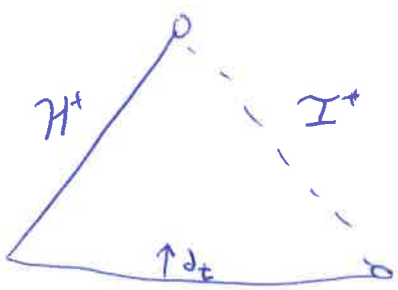


Fig 3



BLACK HOLE STABILITY BACKWARDS AND FORWARDS

MIHALIS DAFERMOS

Slide 4: See figure 1. The initial data is either 2 ended or enters a trapped region. One problem is we don't know which Kerr family we'll approach. Also, the past of future null infinity is the only part we expect to be stable, and we don't even know what that is a priori.

Slide 6: See figure 2. We can ask, are there even *any* spacetimes that have this expected picture? Yes. We can think of H^+ and I^+ as intersecting null hypersurfaces and hope to solve backwards. The theorem on the slide essentially says we can. We can solve backwards to some "initial" hyperboloidal slice.

Slide 12: We probably noticed that even in the poor man's linearization, we already have instability at the extremal case, but in our backwards problem we allowed it. This mystery will be discussed.

Slide 22: Why don't we see the nearby Kerr or Schwarzschild metrics when we perturb Minkowski? Because, in the appropriate Christodoulou-Klainerman norm, they *aren't* close.

Slide 25: See figure 3: Let this be Schwarzschild. The theorem essentially says that if I have finite energy on the null boundaries, then get a unique initial data set.

Slide 26: The reason the forward map fails in Kerr is essentially because ∂_t becomes spacelike.

Slide 27: We need the exponential decay to kill the blue shift so that we get finite energy with respect to N near the horizon for the "initial" data when we solve backwards.

Slide 32: Having a scattering theory near future null infinity would already force exponential decay of energy along future null infinity.

Slide 34: In reality, we really set up finite initial data, solve backwards and then take limit to future null infinity in some sense. Also, we get a uniqueness result in the class of data that exponentially decays, appropriately formulated.

Slide 35: The kind of singularity on H^+ will be a weak null singularity; see Luk's talk on Friday.

In the extremal case, do you still need exponential decay to get this result? The paper conjectures that you could prove that you only need polynomial decay to prove it. You lose blue-shift in the extremal case, which should make it easier.

You could find naked singularities by doing something like this, but you can't get generic solutions by this.

Black hole stability backwards and forwards

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Outline

1. The forward problem: The non-linear stability of Kerr conjecture

2. The backwards problem: Construction of dynamical black holes from scattering data

3. The difficulties and status of the forward problem

4. Proof and lessons from the backwards scattering problem

1. The forward problem: The non-linear stability of Kerr conjecture

Perturbations of a (subextremal) Kerr metric should remain close to *and dynamically approach* the Kerr family in the exterior-to-the-black-hole region.

Conjecture (Stability of Kerr). *Let* $(\Sigma, \bar{g}, K)$ *be a vacuum initial data set sufficiently close to the initial data on a Cauchy hypersurface in the Kerr solution* $(\mathcal{M}, g_{M_i,a_i})$ *for some* <u>*subextremal*</u> *parameters* $0 \leq |a_i| < M_i$. *Then the maximal Cauchy development* $(\mathcal{M}, g)$ *of the data under evolution by the vacuum equations*

$$\mathrm{Ric}(g) = 0$$

possesses a complete null infinity $\mathcal{I}^+$ *such that the metric restricted to* $J^-(\mathcal{I}^+)$ *remains close to for all time and asymptotically approaches a Kerr solution* $(\mathcal{M}, g_{M_f,a_f})$ *in a uniform way with quantitative decay rates, where* $|a_f| < M_f$ *are near* a_i, M_i *respectively.*

Note: $a_i = 0$ <u>will not imply</u> that $a_f = 0$!

2. The backwards problem: Construction of dynamical black holes from scattering data

Theorem (M.D., G. Holzegel, I. Rodnianski). *Given suitable smooth scattering "data" on the horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$, asymptoting to the induced Kerr geometry with parameters $|a| \leq M$, then there exists a corresponding smooth vacuum black hole spacetime asymptotically approaching in its exterior region the Kerr solution with parameters a and M.*

Corollary. *There exist dynamic vacuum black hole spacetimes with no algebraic or geometric symmetries which assymptotically settle down to Kerr.*

Remarks on the statement

1. The set of solutions is parametrized by "a full set" of scattering data. Thus, the class of solutions is "large" in this sense.

2. The assumptions of the theorem will require however that the scattering data decay exponentially *along* $\mathcal{H}^+$, $\mathcal{I}^+$. This is in contrast to the expected behaviour of the "generic" solution of the forward problem, where decay along $\mathcal{H}^+$ and $\mathcal{I}^+$ is expected to be inverse polynomial.

3. Nonetheless, for reasons we shall see, the restriction to exponentially decaying data along $\mathcal{H}^+$ and $\mathcal{I}^+$ is expected to be necessary for the type of formulation as in the Theorem.

3. The difficulties and status of the forward problem

The difficulties of the stability problem can be seen to enter at three levels:

3.1. The "poor man's" linearisation: $\Box_g \psi = 0$.

3.2. The equations of linearised gravity.

3.3 The nonlinear Einstein equations.

3.1 The poor man's linearisation: $\Box_g\psi = 0$ on Kerr

Classical work: WALD 1979, KAY–WALD 1986

Completely solved after intense work in the past 10 years!

... ANDERSSON–BLUE, ARETAKIS, M.D.–RODNIANSKI, LUK, SCHLUE, SHLAPENTOKH-ROTHMAN, TATARU–TOHANEANU

Analogue for $\Lambda \neq 0$ (in the very slowly rotating case):

BONY–HÄFNER, M.D.–RODNIANSKI, DYATLOV, HOLZEGEL, HOLZEGEL–WARNICK, HOLZEGEL–SMULEVICI, VASY, WARNICK

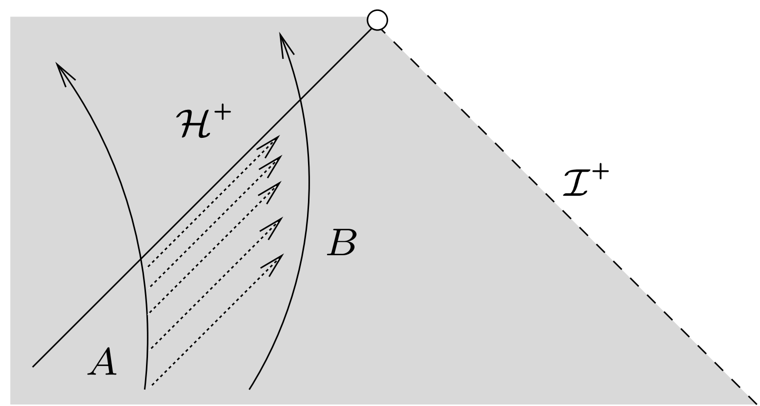
Phenomena

1. Red-shift
2. Superradiance
3. Trapped null geodesics
4. "Low frequency" obstructions
5. Phenomena 1.–4. are strongly coupled as $|a| \to M$.

In fact, the stability result breaks down exactly at $|a| = M$. (Aretakis).

The red-shift

The redshift is classically understood in the geometric optics approximation in terms of signals sent and received by two observers A and B, respectively.



Depends on positivity of surface gravity.

The red-shift is a stability mechanism!

Extremal case $a = M$: <u>The red-shift factor at the horizon vanishes.</u>

3.2 The equations of linearised gravity.

When one linearises the Einstein equations around the trivial solution Minkowski space, say in harmonic coordinates, then each linearised metric component indeed satisfies

$$\Box_g h^{\mu\nu} = 0. \tag{1}$$

When one linearises however, around a nontrivial solution like Kerr, the linearised system has highly non-trivial tensorial structure. This gives rise to additional difficulties not present in (1).

$$\underline{D}_S \frac{\sqrt{g}^{(1)}}{\sqrt{g}_S} = \left(\Omega tr \underline{\chi}\right)^{(1)}$$

$$D_S \frac{\sqrt{g}^{(1)}}{\sqrt{g}_S} = \left(\Omega tr \chi\right)^{(1)} - d\!\!/iv\, b$$

$$\frac{\partial}{\partial u} b^A = 2\Omega_S^2 \left[\left(\eta - \underline{\eta}\right)^\sharp\right]^A$$

$$D_S \left(\Omega tr \underline{\chi}\right)^{(1)} = \Omega_S^2 \left(2 d\!\!/iv \underline{\eta} + 2\hat{\rho} + 4\rho_S \frac{\Omega^{(1)}}{\Omega_S}\right) - \frac{1}{2}\left(\Omega tr\chi\right)_S \left(\left(\Omega tr\underline{\chi}\right)^{(1)} - \left(\Omega tr \chi\right)^{(1)}\right)$$

$$\underline{D}_S \left(\Omega tr \chi\right)^{(1)} = \Omega_S^2 \left(2 d\!\!/iv \eta + 2\hat{\rho} + 4\rho_S \frac{\Omega^{(1)}}{\Omega_S}\right) - \frac{1}{2}\left(\Omega tr\chi\right)_S \left(\left(\Omega tr\underline{\chi}\right)^{(1)} - \left(\Omega tr \chi\right)^{(1)}\right)$$

$$D_S \left(\Omega tr \chi\right)^{(1)} = -\left(\Omega tr\chi\right)_S \left(\Omega tr\chi\right)^{(1)} + 2\omega_S \left(\Omega tr\chi\right)^{(1)} + 2\left(\Omega tr\chi\right)_S \omega^{(1)}$$

$$\underline{D}_S \left(\Omega tr \underline{\chi}\right)^{(1)} = -\left(\Omega tr\underline{\chi}\right)_S \left(\Omega tr\underline{\chi}\right)^{(1)} + 2\underline{\omega}_S \left(\Omega tr\underline{\chi}\right)^{(1)} + 2\left(\Omega tr\underline{\chi}\right)_S \underline{\omega}^{(1)}$$

$$\nabla\!\!\!/_3 \left(\frac{1}{\Omega}\hat{\underline{\chi}}\right) + \frac{1}{\Omega} tr\underline{\chi}\hat{\underline{\chi}} = -\frac{1}{\Omega}\underline{\alpha} \quad , \quad \nabla\!\!\!/_4 \left(\frac{1}{\Omega}\hat{\chi}\right) + \frac{1}{\Omega} tr\chi\hat{\chi} = -\frac{1}{\Omega}\alpha$$

$$\nabla\!\!\!/_3 \left(\Omega\hat{\chi}\right) + \frac{1}{2}\Omega tr\underline{\chi}\hat{\chi} + \frac{1}{2}\Omega tr\chi\underline{\hat{\chi}} = -2\Omega \mathcal{D}\!\!\!/_2^\star \eta$$

$$\nabla\!\!\!/_4 \left(\Omega\underline{\hat{\chi}}\right) + \frac{1}{2}\Omega tr\chi\underline{\hat{\chi}} + \frac{1}{2}\Omega tr\underline{\chi}\hat{\chi} = 2\Omega \mathcal{D}\!\!\!/_2^\star \eta \,.$$

$$d\!\!/iv\,\underline{\hat{\chi}} = -\frac{1}{2}\eta tr\underline{\chi} + \underline{\beta} + \frac{1}{2\Omega_S}\nabla\!\!\!/_A\left(\Omega tr\underline{\chi}\right)^{(1)}$$

$$d\!\!/iv\,\hat{\chi} = -\frac{1}{2}\underline{\eta} tr\chi - \beta + \frac{1}{2\Omega_S}\nabla\!\!\!/_A\left(\Omega tr\chi\right)^{(1)} \,.$$

$$\nabla\!\!\!/_3\underline{\eta} = \frac{1}{2}tr\underline{\chi}\left(\eta - \underline{\eta}\right) + \underline{\beta} \quad , \quad \nabla\!\!\!/_4\eta = -\frac{1}{2}tr\chi\left(\eta - \underline{\eta}\right) - \beta$$

$$D_S\underline{\omega}^{(1)} = -\Omega_S^2\left(\rho^{(1)} + 2\rho_S\frac{\Omega^{(1)}}{\Omega_S}\right)$$

$$\underline{D}_S\omega^{(1)} = -\Omega_S^2\left(\rho^{(1)} + 2\rho_S\frac{\Omega^{(1)}}{\Omega_S}\right)$$

$$\omega^{(1)} = D_S\left(\frac{\Omega^{(1)}}{\Omega_S}\right) \quad , \quad \underline{\omega}^{(1)} = \underline{D}_S\left(\frac{\Omega^{(1)}}{\Omega_S}\right) \quad , \quad \left(\eta + \underline{\eta}\right)^{(1)} = 2\nabla\!\!\!/_A\left(\frac{\Omega^{(1)}}{\Omega_S}\right)$$

$$\nabla\!\!\!/_3\alpha + \frac{1}{2}tr\underline{\chi}\alpha + 2\underline{\hat{\omega}}\alpha = -2\mathcal{D}\!\!\!/_2^\star\beta - 3\hat{\chi}\rho_0$$
$$\nabla\!\!\!/_4\beta + 2tr\chi\beta - \hat{\omega}\beta = d\!\!\!/iv\,\alpha$$
$$\nabla\!\!\!/_3\beta + tr\underline{\chi}\beta + \underline{\hat{\omega}}\beta = \mathcal{D}\!\!\!/_1^\star\left(-\rho^{(1)},\sigma\right) + 3\eta\rho_0$$

$$\nabla\!\!\!/_4\rho^{(1)} + \frac{3}{2}tr\chi\rho^{(1)} = d\!\!\!/iv\,\beta - \frac{3}{2}\frac{\rho_S}{\Omega_S}\left(\Omega tr\chi\right)^{(1)}$$

$$\nabla\!\!\!/_3\rho^{(1)} + \frac{3}{2}tr\underline{\chi}\rho^{(1)} = -d\!\!\!/iv\,\underline{\beta} - \frac{3}{2}\frac{\rho_S}{\Omega_S}\left(\Omega tr\underline{\chi}\right)^{(1)}$$

$$\nabla\!\!\!/_4\sigma + \frac{3}{2}tr\chi\sigma = -cu\!\!\!/rl\,\beta$$
$$\nabla\!\!\!/_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma = -cu\!\!\!/rl\,\underline{\beta}$$
$$\nabla\!\!\!/_4\underline{\beta} + tr\chi\underline{\beta} + \hat{\omega}\underline{\beta} = \mathcal{D}\!\!\!/_1^\star\left(\rho^{(1)},\sigma\right) + 3\eta\rho_0$$
$$\nabla\!\!\!/_3\underline{\beta} + 2tr\underline{\chi}\underline{\beta} - \underline{\hat{\omega}}\underline{\beta} = -d\!\!\!/iv\,\underline{\alpha}$$
$$\nabla\!\!\!/_4\underline{\alpha} + \frac{1}{2}tr\chi\underline{\alpha} + 2\hat{\omega}\underline{\alpha} = 2\mathcal{D}\!\!\!/_2^\star\underline{\beta} - 3\underline{\hat{\chi}}\rho_0$$

New difficulties:

1. Lagrangian structure nonstandard, thus a priori, not clear that there is a conserved or otherwise controlled coercive energy, even in Schwarzschild where ∂_t is causal.

2. Not all degrees of freedom decay, for in particular, linearisation must see nearby Kerr's.

The first problem is by far the biggest.

The second problem already arises in easier problems like wave equation with potential or Maxwell (cf. Blue, Sterbenz–Tataru).

Theorem (M.D.–Holzegel–Rodnianski). *Schwarzschild is linearly stable to gravitational perturbations:*

Solutions of the above system of linearised gravity around Schwarzschild decay polynomially to a solution of linearised Kerr.

See Holzegel's talk on Friday.

3.3 The nonlinear Einstein equations

The difficulties entering at the level of the nonlinearity include of course the familiar difficulties which are already manifest in stability of Minkowski space (Christodoulou–Klainerman).

1. Quadratic nonlinearities in derivatives of the metric, plus quasilinearity. Need special structure to ensure even local existence at $\mathcal{I}^+$.

2. To uncover this structure, need to introduce an elaborate gauge, where wave equations for curvature are coupled with transport and elliptic equations for the connection.

In view, moreover, of the additional difficulties described previously, we could add:

3. How do these difficulties interact with the difficulties of 3.1–3.2? (Holzegel 2010)

4. How does one pick the final parameters a, M?

4. Proof and lessons from the backwards scattering problem

4.1 The scalar wave equation $\Box_g \psi = 0$

DIMOCK-KAY scattering theory for $\Box_g \psi = 0$ on Schwarzschild

Recall the Killing field ∂_t in Schwarzschild.

Let $\mathcal{X}_0$ denote the space of finite energy flux with respect to ∂_t on a slice $t = 0$ of the exterior.

Let $\mathcal{X}_{\mathcal{I}^+}$ denote the space of finite asymptotic energy flux with respect to ∂_t on $\mathcal{I}^+$.

Let $\mathcal{X}_{\mathcal{H}^+}$ denote the space of finite energy flux with respect to ∂_t on $\mathcal{H}^+$. Note that this is highly degenerate!

Then we have:

Theorem (DIMOCK–KAY 1985). *The map $\mathcal{X}_0 \to \mathcal{X}_{\mathcal{I}^+} \oplus \mathcal{X}_{\mathcal{H}^+}$ defined by solving the forward problem and "restricting" to $\mathcal{H}^+$ and $\mathcal{I}^+$ is in fact an isomorphism.*

See also BACHELOT.

This scattering theory, however, unfortunately does not go very far!

On Kerr, the forward map

$$\mathcal{X}_0 \to \mathcal{X}_{\mathcal{I}^+} \oplus \mathcal{X}_{\mathcal{H}^+}$$

is not even well defined.

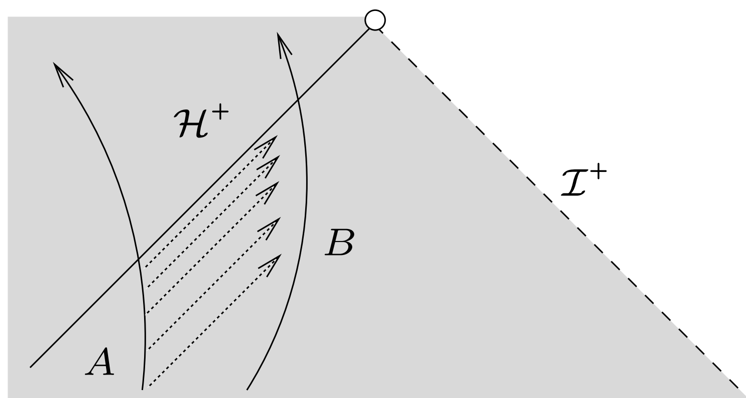
Uniform boundedness is only known for solutions with finite *non-degenerate* positive definite energy.

This is the energy associated with the vector field N related to the red-shift estimate.

Thus the ∂_t-scattering theory is inappropriate even for the scalar wave equation!

N-energy scattering theory for $\Box\psi = 0$ on Schwarzschild/Kerr

The **red-shift** is now a **blue-shift**.



This means that one *must* impose exponential decay along $\mathcal{H}^+$ and $\mathcal{I}^+$.

Once one accepts this obstruction, and imposes such data, then the scattering problem for $\Box_g \psi = 0$ becomes very easy!

One just needs to show that solutions grow at most exponentially when solving backwards. For this, one need only apply the energy identity for N, and Gronwall.

In particular, the difficult of trapped null geodesics, so painful for the forward problem, does not appear.

4.2 Linearised gravity

Recall the characteristic new difficulties in passing from the scalar problem to linearised gravity.

In some sense, the first difficulty (lack of a conserved energy) is not relevant since we are imposing exponential decay and dealing with the N-energy.

On the other hand, the second difficulty remains. Not all degrees of freedom decay, and we need to prevent the non-decaying degrees of freedom from being infinitely **blue-shifted**.

Thus, one must still understand how to “separate out” the degrees of freedom which decay from those that don’t, without destroying the structure of the equations.

4.3 The nonlinear vacuum Einstein equations

Again, recall the difficulties described in the forward problem.

We still have difficulties 1.–3.

(Note that, the blue-shift aside, difficulty 1. again would exclude a scattering theory based solely on the finiteness of the ∂_t flux on null infinity $\mathcal{I}^+$. Even to solve locally around null infinity, one must take weighted estimates, and this will require also decay *along* null infinity, though this obstruction will only be polynomial.)

One slide summary of the proof

We introduce a systematic formulation of a set of

- "renormalised" spin coefficients "Γ", and
- curvature coefficients "ψ"

such that $\Gamma = \psi = 0$ for Kerr (cf. the system used for linearised gravity). These are defined with respect to a null frame adapted to a double null foliation.

The ψ satisfy Bianchi-type equations (hyperbolic) and Γ transport and elliptic equations. The structure of the system is preserved by commutation with respect to an appropriate set of commutation vector fields.

We apply energy estimates to ψ associated with N and with a new hierarchy of weighted vector fields near $\mathcal{I}^+$ capturing peeling. We apply transport estimates to control Γ. The weighted hierarchy also captures the "null condition".

$\Longrightarrow$ these weighted energies grow at most exponentially when solving backwards.

Some technical details

Ambient differential structure.

Approximation by a finite problem.

Prescription of data on null hypersurfaces. Constraints. (See CHRISTODOULOU).

Limit to null infinity. (See CHRISTODOULOU)

Well posedness and inherent loss of derivatives of characteristic initial value problem. (RENDALL, MULLER ZUM HAGEN, CHRISTODOULOU)

Differences of solutions.

Conjecture. *Consider scattering data which decays inverse polynomially along $\mathcal{I}^+$ and $\mathcal{H}^+$. Then one can attach a development spacetime $(\mathcal{M}, g)$, but, for generic such scattering data, $\mathcal{H}^+$ will be singular in the transverse directions.*

cf. Robinson–Trautman spacetimes

This conjecture should not be interpreted as suggesting that generic solutions of the forward problem cannot have polynomial decay! Rather, that one cannot "spot" the solutions of the forward problem arising from smooth data just by looking at the decay on $\mathcal{I}^+$ and $\mathcal{H}^+$.

Kerr-de Sitter? Here, a pure harmonic coordinate approach could work.

cf. also parametrizing solutions from scattering data for asymptotically pure de Sitter (H. Friedrich)