Fig 1

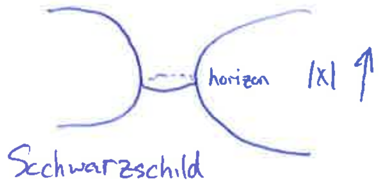


Fig 2

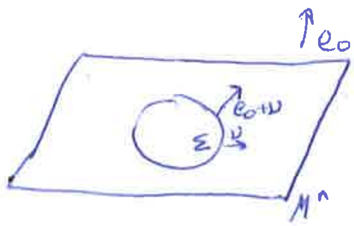


Fig 3

Fig 4

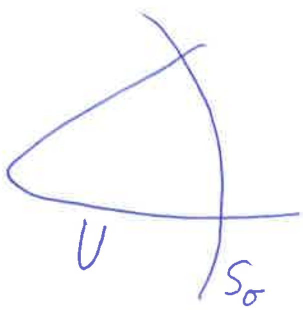


Fig 5

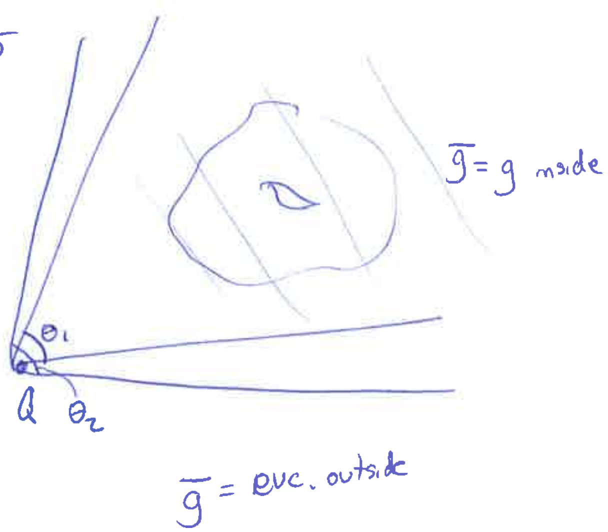
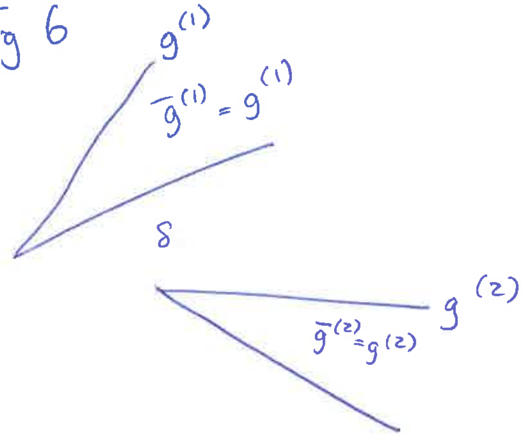


Fig 6



APPROXIMATING GENERAL DATA BY OPTIMALLY FLAT DATA

RICHARD SCHOEN

Minkowski and Schwarzschild solutions slide: See figure 1 for 2-ended picture of the Schwarzschild space.

Outer trapped surfaces slide: See figure 2; the outward pointing future null direction is $e_0 + \nu$ in the figure.

A question coming from the proof of PMT slide: Outside a compact set, say we assume these are graphs. See figure 3.

A further consequence of the positive energy slide: See figure 4.

Localizing in a cone slide: See figure 5.

How do you get smoothness of the metrics at vertex? You deform the transition region near Q in a smooth manner so that there is not sharp vertex.

Construction of n-body solutions slide: I can glue localized solutions together as in figure 6.

If we did this for non-time symmetric, we would still get exactly Euclidean with zero second fundamental form outside the cone.

Could you put, say, Schwarzschild outside this cone? Yes, we can, but we have to perturb inside the cone. I think it should be possible, but I don't know how.

If you did that, maybe you could keep the asymptotics in between the cones? Perhaps.

Can you do this, where you only require a L^2 curvature bound? Yes, this proof allows that.

Approximating general data by optimally flat data

Richard Schoen

Stanford University

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MSRI

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GR Workshop

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November 20, 2013

Plan of Lecture

The lecture will have three parts:

Part 1: Motivation involving the geometry of MOTS

Part 2: Asymptotic behavior and localization of initial data in cones

Part 3: Some features of the proof and applications

Part 1: Motivation: Einstein equations

On a spacetime $\mathcal{S}^{n+1}$, the Einstein equations couple the gravitational field g (a Lorentz metric on $\mathcal{S}$) with the matter fields via their stress-energy tensor T

$$Ric(g) - \frac{1}{2} R \; g = T$$

where Ric denotes the Ricci curvature and $R = Tr_g(Ric(g))$ is the scalar curvature.

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where Ric denotes the Ricci curvature and $R = Tr_g(Ric(g))$ is the scalar curvature.

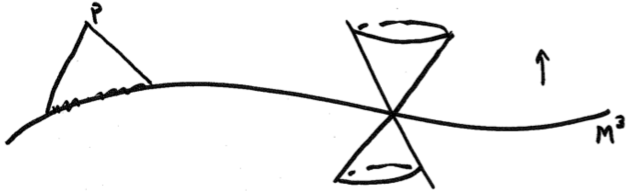
When there are no matter fields present the right hand side T is zero, and the equation reduces to

$$Ric(g) = 0.$$

These equations are called the vacuum Einstein equation.

Initial Data

The solution is determined by initial data given on a spacelike hypersurface M^n in $\mathcal{S}$.



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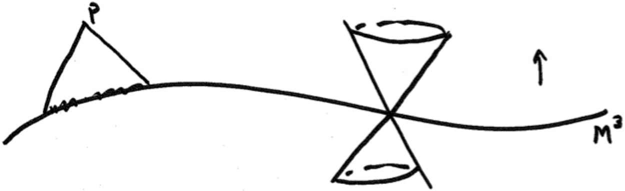

The fields at p are determined by initial data in the part of M which lies in the past of p.

The initial data for g are the induced (Riemannian) metric, also denoted g, and the second fundamental form p. These play the role of the initial position and velocity for the gravitational field. An initial data set is a triple (M, g, p).

The constraint equations for vacuum solutions

It turns out that $n+1$ of the $(n+1)(n+2)/2$ Einstein equations can be expressed entirely in terms of the initial data and so are not dynamical. These come from the Gauss and Codazzi equations of differential geometry.

In case there is no matter present, the vacuum constraint equations become

$$R_M + Tr_g(p)^2 - \|p\|^2 = 0$$
$$\sum_{j=1}^{n} \nabla^j \pi_{ij} = 0$$

for $i = 1, 2, \dots, n$ where R_M is the scalar curvature of M and $\pi_{ij} = p_{ij} - Tr_g(p) g_{ij}$.

The initial value problem

Given an initial data set (M, g, p) satisfying the vacuum constraint equations, there is a unique local spacetime which evolves from that data. This result involves the local solvability of a system of nonlinear wave equations.

The constraint equations with matter present

Using the Einstein equations with matter fields encoded in the stress-energy tensor T together with the Gauss and Codazzi equations, the constraint equations are

$$\mu = \frac{1}{2}(R_M + Tr_g(p)^2 - \|p\|^2)$$
$$J_i = \sum_{j=1}^{n} \nabla^j \pi_{ij}$$

for $i = 1, 2, \ldots, n$ where $\pi_{ij} = p_{ij} - Tr_g(p)g_{ij}$. Here the quantity μ is the observed energy density of the matter fields as seen by an observer moving normal to the spacelike hypersurface, and J is the observed momentum density of the matter. Mathematically the $(n+1)$-vector (μ, J) is gotten by evaluation of T in the direction normal to M.

Energy Conditions

For spacetimes with matter, the stress-energy tensor is normally required to satisfy the **dominant energy condition** $T(u, v) \geq 0$ for any pair u, v of timelike or null vectors. This says that the energy-momentum density $(n+1)$-vector of the matter fields is non-spacelike for any observer. For an initial data set this implies the inequality $\mu \geq \|J\|$.

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In the time symmetric case $(p = 0)$ the dominant energy condition is equivalent to the inequality $R_M \geq 0$.

Asymptotic Flatness

A natural boundary condition for the Einstein equations is the condition of asymptotic flatness. This boundary condition describes isolated systems which are the analogues of finite mass distributions in Newtonian gravity. The requirement is that the initial manifold M outside a compact set be diffeomorphic to the exterior of a ball in R^n and that there be coordinates x in which g and p have appropriate falloff

$$g_{ij} = \delta_{ij} + O_2(|x|^{2-n}),\ p_{ij} = O_1(|x|^{1-n}).$$

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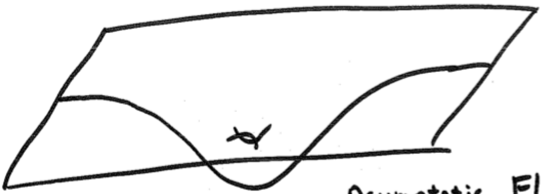

Asymptotic Flatness

Minkowski and Schwarzschild Solutions

The following are two basic examples of asymptotically flat spacetimes:

1) The Minkowski spacetime is R^{n+1} with the flat metric $g = -dx_0^2 + \sum_{i=1}^n dx_i^2$. It is the spacetime of special relativity.

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2) The Schwarzschild spacetime is determined by initial data with $p = 0$ and

$$g_{ij} = (1 + \frac{E}{2|x|^{n-2}})^{\frac{4}{n-2}} \delta_{ij}$$

for $|x| > 0$. It is a vacuum solution describing a static black hole with mass E. It is the analogue of the exterior field in Newtonian gravity induced by a point mass.

ADM Energy and Linear Momentum

For general asymptotically flat initial data sets there is a notion of total (ADM) energy-momentum. These quantities are computed in terms of the asymptotic behavior of g and p. For these definitions we fix asymptotically flat coordinates x and we set
$\pi = p - Tr(p)\, g$.

$$E = \frac{1}{2} \lim_{r \to \infty} \int_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j}) \nu_0^j \, d\sigma_0$$

$$P_i = \lim_{r \to \infty} \int_{|x|=r} \sum_{j=1}^{n} \pi_{ij} \nu_0^j \, d\sigma_0, \qquad i = 1, 2, \ldots, n$$

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These limits exist under quite general asymptotic decay conditions. Generally (E, P) can be thought of as an $(n+1)$-vector in the asymptotic Minkowski space, and for a more general slice in these spacetimes we have the mass $m = \sqrt{E^2 - |P|^2}$.

The positive energy theorem

The positive energy theorem says that $E \geq 0$ whenever the dominant energy condition holds, and that $E = 0$ only if (M, g, p) can be isometrically embedded into the $(n+1)$-dimensional Minkowski space with p as its second fundamental form. In case $p = 0$, the assumption is $R_g \geq 0$, and equality implies that (M, g) is isometric to $\mathbb{R}^n$.

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The problem can be posed in any dimension, and it can be proven in various cases using mean curvature ideas (S & Yau) or using the Dirac operator approach developed by E. Witten. In three dimensions there is a third approach (for $p = 0$) which is the inverse mean curvature flow proposed by R. Geroch and made rigorous by G. Huisken and T. Ilmanen.

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Recently M. Eichmair, L. H. Huang, D. Lee, and S. gave a proof of the stronger inequality $E \geq |P|$ using mean curvature methods.

Outer trapped surfaces

A spacelike surface Σ in a spacetime is **outer trapped** if its area decreases when it is moved in the outward pointing future null direction. The Einstein equations have a focusing effect so that trapped surfaces lead to singularities of the spacetime metric (Penrose).

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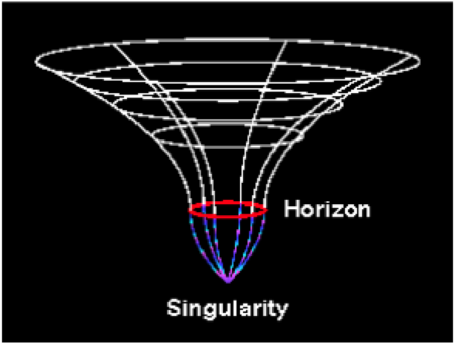

Trapping and mean curvature

If a submanifold Σ is deformed along a vector field X which is orthogonal to Σ, then the logarithmic rate of change of the area is given by the component of the mean curvature in the X direction; that is $\langle \vec{H}, X \rangle$. Thus if we have a spacelike surface $\Sigma \subset M^n \subset \mathcal{S}^{n+1}$ in a spacetime, and we take $X = \nu + e_0$ where e_0 is the future pointing timelike unit normal to M and ν the outward pointing unit normal to Σ in M. We see that the condition that Σ be outer trapped is the condition

$$H + Tr_\Sigma(p) \leq 0$$

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where H is the mean curvature of Σ in M.

Note that if $p = 0$ this is the condition $H \leq 0$.

Minimal surfaces and MOTS

The notion of trapping naturally leads to the notion of a marginally outer trapped surface (MOTS). Such a surface would satisfy $H + Tr_{\Sigma}(p) = 0$, and if it is the boundary between surfaces that are outer trapped and untrapped, it satisfies a stability condition. For $p = 0$ this is the ordinary variational stability of the area functional (second variation nonnegative for all variations).

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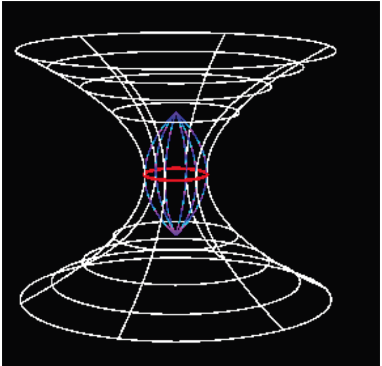
For example the Schwarzschild horizon is a stable minimal surface.



The geometry of MOTS

The constraint equations together with the dominant energy condition tend to force most minimal surfaces and MOTS to be unstable, so they impose strong geometric conditions on a stable MOTS, Σ; for example, a compact Σ is forced to be simply connected ($n = 3$) (Hawking's theorem on the topology of a stationary black hole). Generalized to higher dimensions by G. Galloway and S.. A compact MOTS is Yamabe positive if it is strictly stable; for example, if $\mu > |J|$ at some point on the MOTS.

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G. Galloway wrote a follow-up paper showing that if the MOTS is *outermost* then it is Yamabe positive. Note that this is a global assumption on the MOTS.

A question coming from the proof of PMT

A key ingredient of the mean curvature proof of the PET is the statement that for $n = 3$ there can be no complete asymptotically planar stable minimal surface ($p = 0$) or stable MOTS (general case) provided the dominant energy condition holds **strictly**. For $n \geq 4$ there is a corresponding statement for *strongly* stable MOTS.

A question coming from the proof of PMT

A key ingredient of the mean curvature proof of the PET is the statement that for $n = 3$ there can be no complete asymptotically planar stable minimal surface ($p = 0$) or stable MOTS (general case) provided the dominant energy condition holds **strictly**. For $n \geq 4$ there is a corresponding statement for *strongly* stable MOTS.

It leaves open the question of whether a nontrivial initial data set can contain such a surface. We will give a partial answer to this question in the next part of the talk. It is related to the asymptotic behavior of solutions of the vacuum constraint equations.

A positive result

The following theorem was proven recently by A. Carlotto (arXiv:1310.5118).

Theorem. If (M^3, g, p) is non-trivial, satisfies the dominant energy condition, and is asymptotic to leading order to a slice in the Schwarzschild spacetime, then there is no asymptotically planar stable MOTS. In fact, there is no non-compact stable MOTS with quadratic area growth.

Part 2: Asymptotic behavior

The energy and linear momentum can be shown to exist the under very weak asymptotic decay

$$g_{ij} = \delta_{ij} + O_2(|x|^{-q}),\ p_{ij} = O_1(|x|^{-q-1})$$

for any $q > (n-2)/2$.

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for any $q > (n-2)/2$.

In order to understand the global properties of the Einstein evolution it is important to understand what asymptotic form is reasonable to assume. The positive energy theorem implies that there are no solutions of the constraint equations with compact support.

A further consequence of positive energy

If we let U denote the open subset of M consisting of those points at which the Ricci curvature of g is nonzero, then we have the following. It shows that under reasonable decay conditions the set U must include a positive 'angle' at infinity.

Proposition Assume that (M, g, p) satisfies the decay conditions

$$g_{ij} = \delta_{ij} + O_2(|x|^{2-n}),\ p_{ij} = O_1(|x|^{1-n}).$$

Unless the initial data is trivial, we have

$$\liminf_{\sigma \to \infty} \sigma^{1-n} Vol(U \cap \partial B_\sigma) > 0.$$

Proof of proposition

The energy can be written in terms of the Ricci curvature

$$E = -c_n \lim_{\sigma \to \infty} \sigma \int_{S_\sigma} Ric(\nu, \nu)\ da$$

for a positive constant c_n. If our initial data is nontrivial, then we have $E > 0$, and so for any σ sufficiently large we have

$$E/2 < c_n \sigma \int_{S_\sigma} |Ric(\nu, \nu)|\ da \leq c\sigma^{1-n} Vol(U \cap \partial B_\sigma)$$

where the second inequality follows from the decay assumption.

□

What are good asymptotic forms?

Since it is possible to achieve any chosen pair E, P by a suitably boosted slice in the Schwarzschild, people have assumed that this would be a natural asymptotic form for an asymptotically flat solution of the vacuum constraint equations.

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It was shown by J. Corvino ($p = 0$) and by Corvino and S. (also Chruściel and Delay) that the set of initial data which are identical to a boosted slice of the Kerr (generalization of Schwarzschild) spacetime are dense in a natural topology in the space of all data with reasonable decay.

Localizing in a cone

Let us consider an asymptotically flat manifold (M, g) with $R_g = 0$ and with decay

$$g_{ij} = \delta_{ij} + O_{2,\alpha}(|x|^{-q})$$

where $\alpha \in (0, 1)$ and $(n-2)/2 < q \leq n-2$.

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where $\alpha \in (0, 1)$ and $(n-2)/2 < q \leq n-2$.

In a recent joint work with A. Carlotto we have shown that there is a metric $\bar{g}$ which satisfies $R_{\bar{g}} = 0$ with $\bar{g} = g$ inside a cone based at a point far out in the asymptotic region while $\bar{g} = \delta$ outside a cone with slightly larger angle. Moreover $\bar{g}$ is close to g in a topology in which the energy is continuous, so $\bar{E}$ is arbitrarily close to E. The metric $\bar{g}$ satisfies

$$\bar{g}_{ij} = \delta_{ij} + O_{2,\alpha}(|x|^{-q})$$

provided $q < n-2$.

Where is the energy?

Since there is very little contribution to the energy inside the region where $\bar{g} = g$ and none in the euclidean region, most of the energy resides on the transition region. This shows that one cannot impose too much decay on this region and makes the weakened decay plausible.

Consequences for asymptotically planar stable MOTS

This construction shows that there is an abundance of nontrivial initial data which contain stable minimal surfaces. Since we have constructed solutions which are euclidean in a half space, we can take planes in that half space.

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This construction shows that there is an abundance of nontrivial initial data which contain stable minimal surfaces. Since we have constructed solutions which are euclidean in a half space, we can take planes in that half space.

The construction we have made is limited in the decay which can be arranged, so the question is still open with $|x|^{2-n}$ decay. Some evidence for this was given in the result of A. Carlotto described earlier. He was able to rule out asymptotically planar stable minimal surfaces (and MOTS in the general case) under the assumption that the data is Schwarzschild to leading order at infinity.

Part 3: Some features of the proof and applications

More precise results concerning smoothness and decay:

1) (Hölder spaces) Assume that $g-\delta \in C^{k,\alpha}_{-q} \cap C^{l,\alpha}_{loc}$ with $(n-2)/2 < q < n-2$ and $l \geq k \geq 2$, then for the vertex Q large and fixed angles we have a metric $\bar{g}_Q$ described previously with $\bar{g}_Q - \delta \in C^{k,\alpha}_{-q} \cap C^{l,\alpha}_{loc}$ and the bound

$$\|\bar{g}_Q - \delta\|_{k,\alpha,-q} \leq C = C(g,k,\alpha).$$

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$$\|\bar{g}_Q - \delta\|_{k,\alpha,-q} \leq C = C(g, k, \alpha).$$

2) (Sobolev spaces) Assume that $g - \delta \in W^{k,p}_{-q} \cap W^{l,p}_{loc}$ with $(n-2)/2 < q < n-2$, $l \geq k \geq 2$, $p > 1$, and $kp > n$. It then follows that $\bar{g}_Q - \delta \in W^{k,p}_{-q} \cap W^{l,p}_{loc}$ and

$$\|g - \bar{g}_Q\|_{k,p,-q} \leq \epsilon(|Q|)$$

where $\epsilon(|Q|) \to 0$ as $|Q| \to \infty$.

Outline of proof I

We construct a region Ω which is a cone outside a unit ball centered at Q and on which the transition will occur. We construct a function ϕ which vanishes cleanly on $\partial\Omega$ and is a linear function of the angle outside $B_1(Q)$. We then construct a metric $\tilde{g}$ of the form

$$\tilde{g} = \chi g + (1 - \chi)\delta$$

where $\chi(\phi)$ is a smooth cutoff function which is 1 in a cone of smaller angle and zero outside Ω.

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where $\chi(\phi)$ is a smooth cutoff function which is 1 in a cone of smaller angle and zero outside Ω.

We then seek a solution of the form $\bar{g} = \tilde{g} + h$ where h vanishes in a conical neighborhood of $\partial\Omega$ (and outside Ω) with $R(\bar{g}) = 0$. The equation can be written

$$R(\bar{g}) = R(\tilde{g}) + \tilde{L}h + Q(h) = 0$$

where $\tilde{L}$ is the linearization of the scalar curvature map at $\tilde{g}$. Note that $R(\tilde{g}) = 0$ in a conical region near $\partial\Omega$.

Outline of proof II

We have the formula for the operator

$$\tilde{L}h = \delta\delta h - \Delta_{\tilde{g}}(Tr(h)) - \langle h, Ric(\tilde{g})\rangle$$

where computations are with respect to $\tilde{g}$. The adjoint operator is then

$$\tilde{L}^* u = Hess_{\tilde{g}}(u) - \Delta_{\tilde{g}}(u)\tilde{g} - uRic(\tilde{g}).$$

The composition is given by

$$\begin{aligned}\tilde{L}(\tilde{L}^* u) = (n-1)\Delta(\Delta u) + 1/2(\Delta\tilde{R})u + 3/2\langle\nabla\tilde{R}, \nabla u\rangle \\ + 2\tilde{R}(\Delta u) - \langle Hess(u), Ric(\tilde{g}\rangle\end{aligned}$$

Outline of proof III

We solve the equation

$$\tilde{L}h + Q(h) = f$$

using a Picard iteration scheme in spaces which impose decay of $|x|^{-q}$ at infinity and decay of ϕ^N near $\partial\Omega$ where N is chosen large. The proof involves first showing that $\tilde{L}$ is surjective in such spaces.

Outline of proof III

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The basic estimate which enables us to impose rapid decay near $\partial\Omega$ is

$$\|u\|_{2,-s,\Omega} \leq c\|\tilde{L}^* u\|_{0,-s-2,\Omega}$$

for any $s > 0$ where these are norms in L^2 Sobolev spaces and no boundary condition is imposed on u.

Why do we need $q < n-2$?

We need to show surjectivity of $\tilde{L}$, and this follows from injectivity of $\tilde{L}^*$. The domain of $\tilde{L}^*$ is the dual space of the range of $\tilde{L}$, that is the dual of $H_{0,-2-q}$. This dual space is $H_{0,2+q-n}$ since we have

$$| \int_M f_1 f_2 \; d\mu| \le (\int_M |f_1|^2 |x|^{-n+2(q+2)})^{1/2} (\int_M |f_2|^2 |x|^{n-2(q+2)})^{1/2},$$

and the right hand side is $\|f_1\|_{0,-q-2} \|f_2\|_{q+2-n}$.

Why do we need $q < n - 2$?

We need to show surjectivity of $\tilde{L}$, and this follows from injectivity of $\tilde{L}^*$. The domain of $\tilde{L}^*$ is the dual space of the range of $\tilde{L}$, that is the dual of $H_{0,-2-q}$. This dual space is $H_{0,2+q-n}$ since we have

$$|\int_M f_1 f_2 \, d\mu| \leq (\int_M |f_1|^2 |x|^{-n+2(q+2)})^{1/2} (\int_M |f_2|^2 |x|^{n-2(q+2)})^{1/2},$$

and the right hand side is $\|f_1\|_{0,-q-2} \|f_2\|_{q+2-n}$.

Since $q < n-2$ implies that $s = n - 2 - q > 0$, we can apply the basic estimate to get the injectivity estimate

$$\|u\|_{2,2+q-n} \leq c \|\tilde{L}^* u\|_{0,q-n}.$$

This bound is no longer true if $q \geq n - 2$.

Avoiding derivative loss

A natural way to solve the linear equation $\tilde{L}h = f$ is to look for a solution of the form $\tilde{L}^* u$ and to observe that the operator $\tilde{L}\tilde{L}^*$ is a self adjoint fourth order operator with leading term the bi-harmonic operator. This idea goes back to a 1975 paper of A. Fischer and J. Marsden. It has a drawback in that the lower order terms of the operator $\tilde{L}\tilde{L}^*$ involve four derivatives of $\tilde{g}$, and so the solution $h = \tilde{L}^*(u)$ is two derivatives less smooth than $\tilde{g}$.

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In our setting we can get around this derivative loss by exploiting the fact that our metric $\tilde{g}$ is close to the euclidean metric, and by writing our solution in terms of $L^*(u)$ where L is the euclidean operator. We can then write the linear equation in the form

$$\tilde{L}h = LL^* u + (\tilde{L} - L)(L^* u).$$

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The derivative loss problem in the construction of solutions which are equal to a Kerr slice near infinity was dealt with in a paper of P. Chruściel and E. Delay using a smoothing procedure.

Construction of n-body solutions

Another interesting application of the construction is that it gives a method of 'adding together' initial data. If we have localized solutions we can super-impose them by putting them in disjoint cones. When we do this the energies and linear momenta add up. Since we can approximate a general solution on an arbitrarily large set and in a suitable topology, we can construct n-body initial data with bodies which are far separated. Such a construction was done using the Corvino/S-type construction recently by Chruściel, Corvino, and Isenberg.

Two open questions

This work leaves open the following two questions:

1) If one assumes the Schwarzschild rate of decay $g = \delta + O_2(|x|^{2-n})$ for a nontrivial metric of non-negative scalar curvature, can there be stable asymptotically planar minimal surfaces? There is a similar question for general data with minimal surfaces replaced by MOTS.

2) Can there be an area minimizing asymptotically planar minimal surface in a nontrivial AF space with non-negative scalar curvature?