SOLUTIONS TO THE CONSTRAINT EQUATIONS WITH NON CONSTANT MEAN CURVATURE

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Outline

(1) The conformal method
(2) The Holst-Nagy-Tsogtgerel, Maxwell method
(3) The Dahl-G-Humbert method

1. The conformal method

Constraint equations: For $(M, \tilde{g}, \tilde{K})$, the constraint equations are

$$\mathrm{Scal}_{\tilde{g}} + (\mathrm{tr}_{\tilde{g}} \tilde{K})^2 - |\tilde{K}|^2_{\tilde{g}} = 0$$

$$\tilde{\nabla}^i \tilde{K}_{ij} - \tilde{\nabla}_j (\mathrm{tr}_{\tilde{g}} \tilde{K}) = 0$$

We will assume $n = \dim M \geq 3$, $N = \frac{2n}{n-2}$ and $N - 2 = \frac{4}{n-2}$ which is the nice exponent for making the conformal transformation.

We set $\tilde{g} = \phi^{N-2} g$ and $\tilde{K} = \frac{\tau}{n}\tilde{g} + \phi^{-2}(\sigma + LW)$, where g is a Riemannian metric on M. Also, $\tau : M \to \mathbb{R}$, a function, and σ, a symmetric traceless 2-tensor such that $\nabla^i \sigma_{ij} = 0$, are given. We are solving for $\phi : M \to \mathbb{R}^*_+$, a conformal factor and W, a 1-form. Here, LW is the conformal Killing operator,

$$LW_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \nabla^k W_k g_{ij}.$$

Using these, we can rewrite the constraint equations as

$$-\frac{4(n-1)}{n-2} \Delta \phi + \mathrm{Scal}^g \phi + \frac{n-1}{n} \tau^2 \phi^{N-1} = |\sigma + LW|^2 \phi^{-N-1}$$

$$-\frac{1}{2} L^* LW = \frac{n-1}{n} \phi^N d\tau.$$

The first equation is the Lichnerowicz equation. The second one is called the vector equation. Together they are called the conformal constraint equations. Here, L^* is the formal L^2 adjoint of L, and $(-\frac{1}{2} L^* LW)_j = \nabla^i LW_{ij}$.

If we take the trace of $\tilde{K}$ with respect to $\tilde{g}$, we get τ. This justifies the name mean curvature for τ. If $d\tau$ is 0, then the equations decouple. The vector equation is easy to solve (usually just 0 in this case) and then we can solve the Lichnerowicz equation.

Solutions to this system are well understood when τ is constant and, by perturbation arguments, when τ is close to constant, i.e. when the coupling is small.

What happens when $d\tau$ is not small? Suppose I can define λ as the magnitude of ϕ. Then the vector equation gives that W has order λ^N. In the Lichnerowicz equation, we see that both the τ^2 term and the $|\sigma + LW|^2$ are of order λ^{N-1}.

In elliptic equations it is important to get a priori estimates. But here, just looking at the order of magnitude, the two dominant terms have the same order of magnitude but opposite signs, so there is no a priori bound on ϕ.

2. The HNT-M method

What if we choose λ small?

Theorem 2.1. *Assume that M is compact and that $Y(g) > 0$. (This is the Yamabe invariant,*

$$Y(g) = \inf_{u \in C^1, u \not\equiv 0} \frac{\int \frac{(4(n-1)}{n-2} |\nabla u|^2 + Scal^g u^2)}{\left(\int u^N\right)^{2/N}}.)$$

Then if $\sigma \not\equiv 0$ and is sufficiently small, then the conformal constraint equations have a solution.

Proof. Set $\phi = \lambda \hat{\phi}$, $W = \lambda^N \hat{W}$ and $\sigma = \lambda^{\frac{N}{2}+1} \hat{\sigma}$. We rewrite the conformal constraint equations as (ignoring coefficients)

$$-\Delta \hat{\phi} + \mathrm{Scal} \hat{\phi} + \frac{n-1}{n} (\lambda^{N-2} \tau^2) \hat{\phi}^{N-1} = |\hat{\sigma} + \lambda^{\cdots} L\hat{W}|^2 \hat{\phi}^{-N-1}$$

$$-\frac{1}{2} L^* L \hat{W} = \hat{\phi}^N d\tau$$

In this system, let $\lambda \to 0$. The first equation becomes

$$-\Delta \hat{\phi} + \mathrm{Scal} \hat{\phi} = |\sigma|^2 \hat{\phi}^{-N-1}$$

which is just the Lichnerowicz equation with $\tau = 0$, and $\sigma + LW$ replaced by σ. This system thus decouples for $\lambda = 0$. We can then use the implicit function theorem to get a solution for λ small. □

This theorem is limited to the case where $Y(g) > 0$, and we construct solutions that are very close to zero, and so we might want to look at other types of manifolds. It is not known how to do this for noncompact cases. [Well, it is. It should be posted on ArXiV soon for the AE case.]

3. The DGH method

What if λ is very large? Looking at the Lichnerowicz equation, we would expect the τ^2 and $|\sigma + LW|^2$ terms should dominate, and we should get something like

$$\frac{n-1}{n} \tau^2 \phi^{N-1} \simeq |LW|^2 \phi^{-N-1}.$$

We can then plug this into the second equation to get a nontrivial solution to

$$-\frac{1}{2}L^*LW = \sqrt{\frac{n-1}{n}}|LW|\frac{d\tau}{|\tau|},$$

which we call the limit equation. Heuristically, if this equation has no nontrivial solution, then λ cannot become very large, and so we get an a priori bound for ϕ and W which we can then use to solve this system.

Theorem 3.1 (DGH). *Assume that M is compact and that $\sigma \not\equiv 0$ if $Y(g) \geq 0$. Also, assume that $\tau \geq \tau_0 > 0$, for τ_0 arbitrarily small. Then if the limit equation*

$$-\frac{1}{2}L^*LW = \alpha\sqrt{\frac{n-1}{n}}|LW|\frac{d\tau}{\tau}$$

has no nonzero solution for any $\alpha \in (0,1]$, then the set of solutions to the conformal constraint equations is nonempty and compact.

This requires no restriction on the sign of the Yamabe invariant, but we need the mean curvature is bounded from below. This method also works for AH manifolds (G-Sakovich) and AE manifolds (Dilts-G-Isenberg) (where we need that $\tau \geq \epsilon r^{-\alpha}$ with $\alpha \in (1, n-1)$ [He wrote it incorrectly on the board, but this is correct].

To use this theorem, we still need to prove the limit equation has no solutions.

Example: If (M,g) is Einstein with $\mathrm{Ric} = -(n-1)g$, then the limit equation has no solution provided that $\|d\tau/\tau\|_{L^\infty} < \sqrt{n}$. Recently we [Gicquaud and Ngo Quoc Anh, his post-doc] have come up with a simplified proof based on the Leray-Schauder theorem: Take X a Banach space, $F : X \to X$ continuous such that $F(B)$ is compact for any bounded set B. Then if the set $\{(x,\lambda) : s = \lambda F(x)\}$ is bounded, then F has a fixed point, and the set of fixed points is compact.

Proof of DGH. We choose $X = L^\infty(M, \mathbb{R})$. We define

$$F : X \to X$$

by decomposing it into pieces $X \to W^{2,p} \to C^1 \to X$ where the maps are $\phi \mapsto W \mapsto W \mapsto \psi$ where the first map is solving the vector equation, the second is the compact embedding [using Rellich-Kondrachov] and the third is solving the Lichnerowicz equation using the given W. We can then show that F is continuous and compact. Thus all we have left is to prove that the given set is bounded, i.e. the set of λ-fixed points.

Assume that the set of λ-fixed points is unbounded. Then we get a sequence (λ_i, ψ_i) such that $\|\psi_i\|_{L^\infty} \to \infty$. Set $\phi_i = F(\psi_i) = \psi_i/\lambda_i$. Set $\gamma_i = \|\phi_i\|_{L^\infty}$. Let $\hat{\phi}_i = \frac{1}{\gamma^i}\phi_i$ and $\hat{W}_i = \frac{1}{\gamma_i^N}W_i$ and $\hat{\sigma} = \frac{1}{\gamma_i^N}\sigma$. We get

$$\frac{1}{\gamma_i^{N-2}}(-\Delta\hat{\phi}_i + R\hat{\phi}_i) + \frac{n-1}{n}\tau^2\hat{\phi}_i^{N-1} = |\hat{\sigma}_i + L\hat{W}_i|^2\hat{\phi}_i^{-N-1}$$

$$-\frac{1}{2}L^*L\hat{W}_i = \frac{n-1}{n}\hat{\phi}_i^N d\tau$$

Clearly we have $\hat{\phi}_i^N$ bounded in L^∞ because of the rescaling and so the $\hat{W}_i$ are bounded in $W^{2,p}$, and so we can extract a subsequence that converges in C^1. Thus we only need to prove that $\hat{\phi}_i \to \hat{\phi}_\infty$ where

$$\hat{\phi}_\infty^N = \sqrt{\frac{n-1}{n}}\frac{|L\hat{W}_\infty|}{|\tau|}.$$

Choose $\epsilon > 0$ and $\hat{\phi}_\pm \in C^2$ such that $\hat{\phi}_\infty - \epsilon \leq \hat{\phi}_- \leq \hat{\phi}_\infty - \epsilon/2$ and $\hat{\phi}_\infty + \epsilon/2 \leq \phi_+ \leq \hat{\phi}_+ + \epsilon$. If γ_i is large enough, the $-\Delta + R$ term completely disappears from the Lichnerowicz equation, and so $\hat{\phi}_\pm$ are sub and supersolutions for the Lichnerowicz equation for large enough γ_i. Thus $\hat{\phi}_- \leq \hat{\phi}_i \leq \hat{\phi}_+$ for large enough i. Thus $\hat{\phi}_i \to \hat{\phi}_\infty$. □

It looks like there's an improvement. Do you get a uniqueness result from using the implicit function theorem? No, because you could have another solution that goes to infinity as $\lambda \to 0$.