Fig 1:

Fig 2

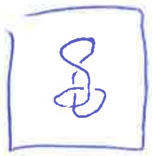


Fig 3

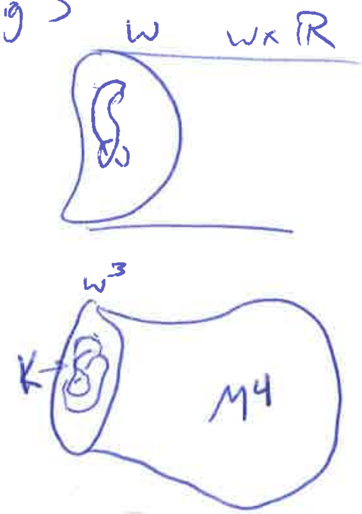


Fig 4

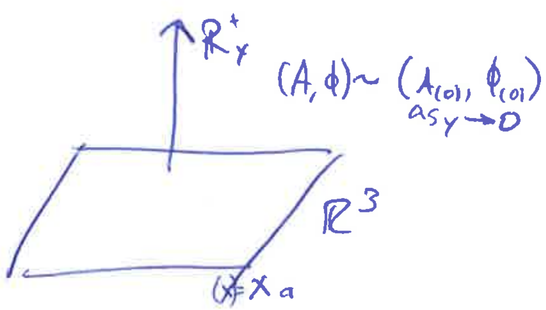
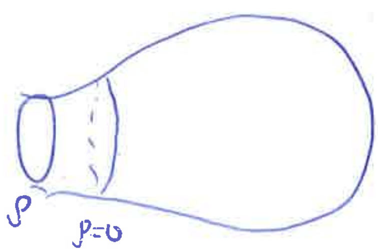


Fig 5



THE NAHM POLE BOUNDARY CONDITIONS FOR THE KW EQUATIONS

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This talk will be on gauge theory, not relativity.

I will be using gauge theory to study topological invariants. Witten proposed an interesting program a few years ago; we're trying to make it work.

For any W^3, a compact manifold, there is a Casson invariant. The study of this led to Floer theory. We start with a principle G bundle E over W, where G is a simple group. We then look at connections A on E, and study the Chern-Simons functional

$$CS(A) = \frac{1}{4\pi}\int_W \mathrm{tr} A \wedge dA + \frac{2}{3} A \wedge A \wedge A.$$

Let $F_A = \partial A + A \wedge A$. A is a 1 form, but it is really shorthand for some reference connection plus a one form, i.e. $\nabla^0 + A$. The flat connection is where the "curvature" one form is zero.

We can take the gradient of the CS functional on $W \times \mathbb{R}$. See fig 1. We can then use it to make a gradient flow. The Euler invariant from the flow lines gives the Casson invariant.

Survey article: Khovanov homology and Gauge theory by Witten (2012).

This is a proposed scheme to understand knot invariants using gauge theory. See figure 2.

If you look at the gradient flow equations, $\partial A/\partial t = \nabla CS(A)$, they are equivalent to $F_\Lambda^+ = 0$, i.e. the self dual equations.

Witten said there is a gauge theory on manifolds like $W \times \mathbb{R}_y^+$. "Firebranes and Knots" gives another introduction for this topic, more physically motivated.

First, we replace the group with its complexification, $G \mapsto G_{\mathbb{C}}$, (like $SU(n) \mapsto Sl(n, \mathbb{C})$), and send $A \mapsto \mathcal{A} = A + i\phi$. Then consider $CS(\mathcal{A})$. We choose a parameter α, then take $\Re(e^{i\alpha} CS(\mathcal{A}))$, which is now a real valued function. We also take $\partial_y = -\nabla\Re(e^{i\alpha} CS(\mathcal{A}))$. This recaptures idea of gradient flow, which doesn't work out so nicely if the function was complex.

These equations are invariant under G, but not $G_{\mathbb{C}}$. So, we look at solutions lying in the zero set of the moment map. We put an extra equation to get this: $d_A \star \phi = 0$.

The KW (Kapuscin-Witten) equations (also by Haydys) are $F_A - \phi\wedge\phi - \star\partial_A\phi = 0$ and $\partial_A \star \phi = 0$. Typically we want to add a gauge condition to get these to be elliptic, $\partial \star (A - A_{(0)}) = 0$, for some reference connection $A_{(0)}$. We should think of A as a 1 form, and $\star\phi$ as a 3 form. Then this is really just some lower order perturbation of $d + d^*$, and so they are elliptic.

See Fig 3. We have a knot on the boundary, and can consider $W \times \mathbb{R}$ or a 4-manifold with a 3-manifold boundary. We look for fields (A, ϕ) such that $\phi \sim 1/y$ on $W \setminus K$ and ϕ has some extra singularity along K. Classical theory won't work quite right because it is somewhat singular at the boundary. Today, we'll talk about the vacuum case, where the knot is empty. This would correspond to the case where the Khovanov homology is zero.

Theorem 0.1 (M-Witten)**.** *(1) When K is empty, this is an elliptic boundary problem (in the uniformly degenerate calculus).*
(2) Ind$(DKW) = -(\dim \mathfrak{g})\chi(M)$
(3) Uniqueness theorem: If (A, ϕ) is a solution on $\mathbb{R}^3 \times \mathbb{R}^+$ and these are "asymptotically Nahm" at infinity, then (A, ϕ) is the model Nahm solution.

Model solution: This lives in $\mathbb{R}^1_y$. Let $\mathfrak{g}$ be the Lie algebra of G. We pick $t_1, t_2, t_3 \in \mathfrak{g}$ such that they generate $SU(2)$. Let $a = 1, 2, 3$ and $i = 0, 1, 2, 3$. Thus we have $[t_1, t_2] = t_3$ and similar cyclic permutations. There are lots of possible choices for this. Let $A_{(0)} = \partial$. Do this on $\mathbb{R}^3 \times \mathbb{R}^1_y$. Let $\phi = \phi_{(0)} = \frac{\sum t_a dx^a}{y}$. If we plug this into the equations, we can see that these are solutions, thanks to the commutation relationships.

Definition 0.2. On M, $\partial M = W$, (A, ϕ) satisfy Nahm-pole boundary conditions provided $A = A_{(0)} + o(y^0)$, lower order stuff, and $\phi = \frac{\sum t_a dx^a}{y} + o(y^{-1})$.

There is some rigidity for ϕ. It looks like you're choosing some frame, but this definition is really natural. In general, we can pick a representation $\rho : SU(2) \to \mathfrak{g}$. We pick a standard basis $\tau_a \mapsto t_a$. We want to make sense of $\phi_\rho = \sum t_a dx^a \in ad(E) \otimes T^*W$. If I feed in an orthonormal basis, I get $\phi_\rho(e_a) = t_a$. If I pick a different basis, I get different t_a. Thus $\phi_\rho \in C^\infty(W, ad(E) \otimes T^*W)$. So, this is really a rigid condition. I have choices, but they're only discrete, so I can't deform the top order part of ϕ.

What about A? If I look at these in the equations, I have a $1/y^2$ term and get Nahm's equation. The $1/y$ term says you have a $\partial_{A_{(0)}} \phi_\rho = 0$. If you decode this, this means the connection on the image of ϕ_ρ is the Levi-Civita connection of W. This is the rigidity of the leading part of A.

Uniqueness theorem: See fig 4. Also, as $|(\vec{x}, y)| \to \infty$ we also have $(A, \phi) \sim (A_{(0)}, \phi_{(0)})$ as in the figure. We can conclude that $(A, \phi) \equiv (A_{(0)}, \phi_{(0)})$, a Nahm Solution.

If this wasn't true, then I would have something interesting going on, then I could scale it to $y = 0$, and it could bubble. This says that there can't be any bubbles on the boundary. It is possible they could bubble in the center. See Gagliardo-Uhlenbech, Taubes for more on this problem. The intuition is that there is no interior bubbling.

How to prove that the solutions are equivalent? We use a Weitzenboch formula. Let

$$V_{ij} = F_{ij} - [\phi_i, \phi_j] + \epsilon_{ij}^{hl} D_h \phi_l,$$

where F_{ij} is the 2 form component of the curvature. Let $V_0 = \partial_A^* \phi$.

$$\int_M -\mathrm{tr}\left(V_{ij}V^{ij} + (V^0)^2\right) = \int -\mathrm{tr}\left(\frac{1}{2}F_{ij}F^{ij} + \sum (D_a\phi_b)^2 + \sum (D_i\phi_y)^2 \right.$$
$$\left. + ([\phi_y, \phi_a])^2 + \sum (d_y\phi_a + \frac{1}{2}\epsilon_{abc}[\phi_b, \phi_c])^2\right) + \int_W \epsilon^{abc}\mathrm{tr}(\phi_a F_{bc})$$

and other boundary terms. Parts of this disappear if we have solutions. The last term in the main integral vanishes for Nahm solutions, and so it vanishes for solutions near the boundary. This would show uniqueness if the boundary term vanished.

At a formal level, this just works. We've assumed that $A = A_{(0)} + o(1)$ and $\phi = \sum t_a dx^a/y + o(1/y)$, and so we can't say anything about the boundary terms with just this. We need to improve this into an actual expansion, something like $A = A_{(0)} + ya_1 + \cdots$. In the best case we have $\phi = \sum stuff + \phi_1 y + \cdots$. In this case, the boundary terms cancel out, and so it works.

In general, there are more complicated expansions. They aren't so hard, but they do involve powers of $y^{1/2}$. It takes a bit of work to make it all work, but it does. Why should you expect such an expansion? The whole point is that this is not a standard elliptic problem.

If I take $DKW|_{(A_{(0)}, \phi_{(0)})} = \mathcal{L} \simeq \partial_y + \sum B_a \partial_{x_a} + \frac{1}{y}B_0$, where we are restricting to an approximate Nahm pole solution. $\mathcal{L}$ is a linear operator, the B's depend on x and y. I could look for solutions of the form $y^\lambda a$ and $y^\lambda \phi$. If I put these into the linearized equations, I get, among other things,

$$\lambda a_a + \frac{1}{y}([t_a, \phi_y] - \epsilon_{abc}[t_b, a_c]) = 0$$

and $\lambda \phi_y + [t_a, \phi_a] = 0$. We have to attack this with representation theory. We have look at $\rho(SU(2)) \subset \mathfrak{g}$, calculate indicial roots, and then we get the y's to the $1/2$.

We use $\mathcal{L}$ to produce a parametrix G, an approximate inverse, and then try understand its sharp mapping properties.

What happens with knots? Look at $Ind(\mathcal{L})$. See fig 5. If I take this, then the index on the left part plus the index of the right part give the index of the whole operator. The right part gives the Euler characteristic (with a sign). On the left part, we get index zero thanks to the splitting.

Consider Σ^2 with a point singularity. There is a good model solution at that point. There's a severe singularity at the point, but there's still a good model.

In my experience, the big issue is boundary regularity. There are two boundaries, overall and near the knot. If we can control that, we can get the overall control we want.