Fig 1

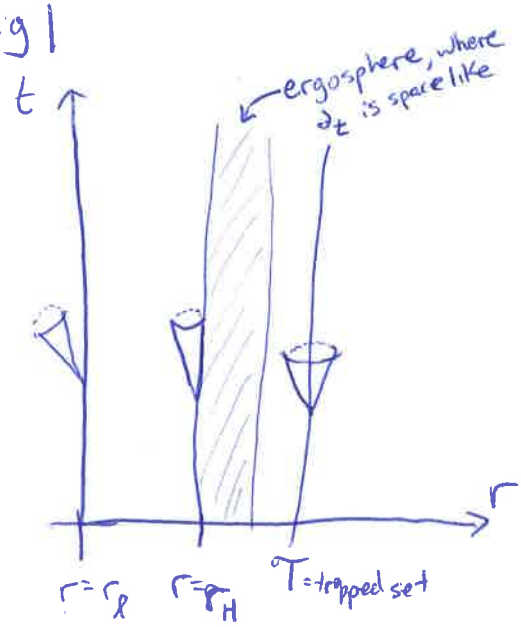


Fig 2

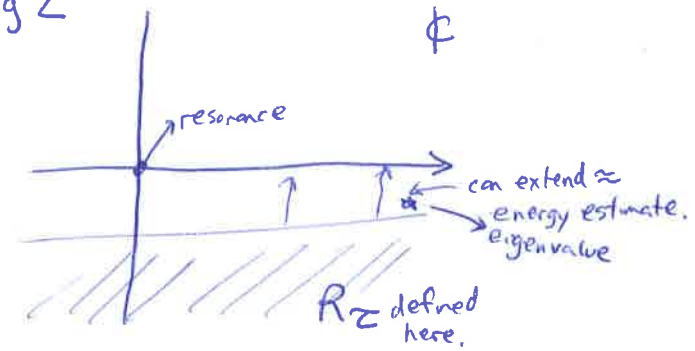
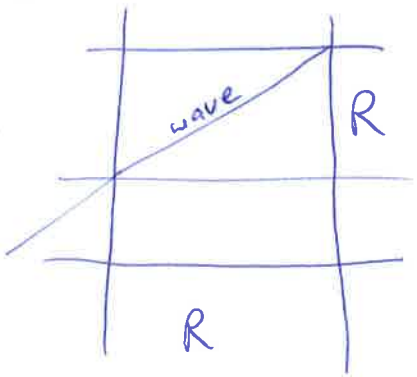


Fig 3



DECAY OF SCALAR AND ELECTROMAGNETIC WAVES ON BLACK HOLE SPACE-TIMES

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Slide 4: See figure 1.
Slide 7: See figure 2.
Slide 8: See figure 3.
Slide 13: $\Box + V$, where V is order of r^{-3} is enough to produce this.

Decay of scalar and electromagnetic waves on black hole space-times

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MSRI, Nov. 2013

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Outline.

1. Asymptotically flat space-times
2. Decay estimates for scalar waves
3. Local energy decay
4. Electromagnetic waves

Asymptotically flat nontrapping space-times

Domain: $\mathbb{R}^{3+1}$.

Lorenzian metric: $g = g_{\alpha\beta} dx^\alpha dx^\beta$.

Space-like foliation: $t = const$, normal $N = \nabla t$, space-like.

Asymptotically flat: $g = e + O(r^{-\epsilon})$.

Stationary: Killing field $X = \partial_t$, time-like.

Nontrapping: all null geodesics escape to infinity.

More general:

- small time dependent perturbation thereof,
- potentials,
- magnetic fields

Asymptotically flat black hole space-times

Domain: $\mathbb{R}^{3+1} \supset \mathcal{M} = \{r > r_0\}$,

Lorenzian metric: $g = g_{\alpha\beta} dx^\alpha dx^\beta$.

Space-like foliation: $t = const$, normal $N = \nabla t$, time-like.

Asymptotically flat: $g = m + O_{rad}(1/r) + O(1/r^2)$.

Event horizon: $\mathcal{H} = \{r = r_{\mathcal{H}}\}$, $r_{\mathcal{H}} > r_0$.

Null generator: $L = \nabla r$, tangent to $\mathcal{H}$, $\nabla_L L = \sigma L$.

Trapped set: $\mathcal{T} \subset \{r > r_{\mathcal{H}}\}$, compact.

Killing field: $X = \partial_t$, time-like outside a compact set.

More general: small time dependent perturbation thereof, etc.

Scalar waves

Inhomogeneous wave equation:

$$\Box_g u = f, \qquad u[0] := (u(0), Nu(0)) = (u_0, u_1).$$

Also with magnetic field and/or potential.

Energy momentum tensor:

$$T_{\alpha\beta} = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} \partial^\nu u \partial_\nu u,$$

$$\nabla^\alpha T_{\alpha\beta} = 0, \qquad \nabla^\alpha (T_{\alpha\beta} X^\beta) = 0.$$

Conserved energy:

$$E = \int T(X, N) dV.$$

Positive definite in nontrapping case, positive definite outside a compact set in black hole case.

Decay estimates for wave equations

Equation:

$$(\Box_g + V)u = f, \qquad u[0] = (u_0, u_1).$$

Decay estimates for linear waves:

- Uniform energy bounds
- Local energy decay
- Strichartz estimates
- Pointwise decay

Goals:

- Do such properties hold for physically relevant space-times ?
- Characterization in terms of spectral properties
- Stability with respect to (time dependent) perturbations

Possible obstructions:

- Low frequency: eigenvalues, resonances
- High frequency: trapping

Uniform energy bounds and the resolvent

$$(E) \qquad \|u[t]\|_{\dot{H}^1 \times L^2} \lesssim \|u[0]\|_{\dot{H}^1 \times L^2}.$$

To define the resolvent take a time Fourier transform

$$\Box_g u = f \longrightarrow P_\tau \hat{u}(\tau) = \hat{f}(\tau) \longleftrightarrow \hat{u}(\tau) = R_\tau f(\tau)$$

In product case, $g = -dt^2 + g_0$, $\quad R_\tau = (\Delta_{g_0} + \tau^2)^{-1}$.
A-priori we have exponential bounds

$$\|u[t]\|_{\dot{H}^1 \times L^2} \lesssim e^{Mt} \|u[0]\|_{\dot{H}^1 \times L^2}.$$

so resolvent is well defined and holomorphic for $\Im\tau < -M$.

Proposition

Uniform energy bounds are equivalent to the resolvent bound

$$\|R_\tau\|_{L^2 \to \dot{H}^1} \lesssim |\Im\tau|^{-1}, \qquad \Im\tau < 0$$

Eigenvalues (Must be on imaginary axis in product case.):

$$P_\tau u = 0, \qquad \Im\tau < 0$$

Local energy decay in Minkowski space-time

$$\Box\phi = 0 \qquad \text{in } \mathbb{R}^{n+1}, \qquad \phi[0] = (\phi_0, \phi_1).$$

Local energy decay (also known as *Morawetz estimates*):

$$\|\nabla_{x,t}\phi(x,t)\|_{L^2(\mathbb{R}\times B_R)} \lesssim R^{\frac{1}{2}}\|\nabla_{x,t}\phi(x,0)\|_{L^2}.$$

Heuristics: A speed 1 wave spends at most $O(R)$ time inside B_R. Morawetz's proof uses the positive commutator method. If P and Q are selfadjoint, respectively skewadjoint operators then

$$2\Re\langle P\phi, Q\phi\rangle = \langle [Q,P]\phi, \phi\rangle$$

Apply this with

$$P = \Box, \qquad Q = \partial_r + \frac{n-1}{2r},$$

to obtain

$$\|r^{-\frac{1}{2}}\nabla\!\!\!/\phi(x,t)\|_{L^2} + \|\phi(0,t)\|_{L^2} \lesssim \|\nabla_{x,t}\phi(x,0)\|_{L^2}, \qquad n = 3$$

The local energy norms

At the L^2 level we set

$$\|u\|_{LE} = \sup_k \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2(\mathbb{R}\times A_k)}, \qquad A_k = \{|x| \approx 2^k\} \times \mathbb{R}$$

We also define its H^1 counterpart, as well as the dual norm

$$\|u\|_{LE^1} = \|\nabla u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE} \quad \|f\|_{LE^*} = \sum_k \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R}\times A_k)}$$

Sharp formulation of local energy decay:

$$(LE) \qquad \|u\|_{LE^1} + \|\nabla u\|_{L^\infty L^2} \lesssim \|\Box u\|_{LE^* + L^1 L^2} + \|\nabla u(0)\|_{L^2}$$

Proposition

Assume uniform energy bounds. Then local energy decay is equivalent to the uniform resolvent bound

$$\|R_\tau f\|_{LE^1_0} \lesssim \|f\|_{LE^*_0} \ , \qquad \Im \tau \leq 0$$

Embedded resonances

These are obstructions to the resolvent local energy decay estimate,

$$\|R_\tau f\|_{LE^1_0} \lesssim \|f\|_{LE^*_0} \ , \qquad \Im\tau \leq 0$$

On real axis R_τ is defined as the limit as $\Im\tau \to 0$. This implies the outgoing radiation condition

$$r^{-\frac{1}{2}}(\partial_r - i\tau)u \in L^2, \qquad u = R_\tau f.$$

Definition

$u \in LE^1_0$ is an embedded resonance if it satisfies the outgoing radiation condition and $P_\tau u = 0$.

Local energy decay in geometries with trapping

Example: Schwarzschild space-time, with trapped set = all null geodesics tangent to the photon sphere $r = 3M$.

Redeeming feature: hyperbolic flow around trapped null geodesics.

Heuristics: frequency λ waves will stay localized up to time $\log \lambda$ (Ehrenfest time) near the trapped set, then disperse.

Consequence: $|\log \lambda|^{\frac{1}{2}}$ loss in (LE) at frequency λ on trapped set.

Modified local energy norm has log losses on the trapped set,

$$LE^1 \subset LE^1_{\mathcal{T}}, \qquad LE^*_{\mathcal{T}} \subset LE^*$$

with equality away from $\mathcal{T}$. Local energy decay:

$$(LE) \qquad \|u\|_{LE^1_{\mathcal{T}}} + \|\nabla u\|_{L^\infty L^2} \lesssim \|\Box u\|_{LE^*_{\mathcal{T}} + L^1 L^2} + \|\nabla u(0)\|_{L^2}$$

Similar modification in resolvent bounds.

Strichartz estimates

Range of indices in $3+1$ dimensions:

$$2 < p \leq \infty, \qquad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$$

Direct estimate for $\Box_g u = 0$:

$$\| |D_x|^{-\rho} \nabla u \|_{L^p L^q} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2}, \quad \rho = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}$$

Inhomogeneous estimate for $\Box_g u = f$, $u[0] = 0$

$$\|\nabla u\|_{L^\infty L^2} \lesssim \| |D_x|^{\rho} f \|_{L^{p'} L^{q'}}$$

Retarded estimate for $\Box_g u = f$, $u[0] = (u_0, u_1)$:

$$\| |D_x|^{-\rho} \nabla u \|_{L^p L^q} + \|\nabla u\|_{L^\infty L^2} \lesssim \|f\|_{|D_x|^{-\rho} L^{p'} L^{q'} + L^1 L^2} + \|u[0]\|_{\dot{H}^1 \times L^2}$$

Pointwise decay estimates (Price Law)

Set-up at infinity:

$$g = m + O_{rad}(r^{-1}) + O(r^{-2}), \qquad V = O_{rad}(r^{-3}) + O(r^{-4}).$$

(Improved) Price Law:

$$|u(t,x)| \lesssim \frac{1}{\langle t\rangle\langle t-|x|\rangle^2}\|\nabla u(0)\|_{H^{m,k}},$$

$$|\partial_t u(t,x)| \lesssim \frac{1}{\langle t\rangle\langle t-|x|\rangle^3}\|\nabla u(0)\|_{H^{m,k}}.$$

$$|\partial_x u(t,x)| \lesssim \frac{1}{\langle r\rangle\langle t-|x|\rangle^3}\|\nabla u(0)\|_{H^{m,k}}.$$

Remark

When true, the above decay rates are sharp due to the contribution of the leading order radial terms in the metric or potential.

Local Energy Decay as a central concept

Connection to Strichartz estimates:

Theorem (Metcalfe-T. '07 (nontrapping, nonstationary))

Assume that uniform energy bounds and local energy decay hold. Then the Strichartz estimates hold.

Idea: Outgoing parametrix with good pointwise decay estimates. The same method applies in the black hole setting, provided one has only hyperbolic trapping, and a good result near the trapped set $\mathcal{T}$ (e.g. Burq - Guillarmou-Hassell).

Connection to pointwise decay estimates:

Theorem (T. '09 (stationary), Metcalfe-T.-Tohaneanu '11 (non-stat.))

Assume that uniform energy bounds and local energy decay hold. Then the pointwise decay bounds hold (Price's Law).

Idea: Combine Klainerman's vector field method near the light cone with local energy decay inside the cone, reiterate.

Local energy decay in the nontrapping case

Theorem (Metcalfe-T.'08 (nonstationary))

Local energy decay holds if g is a small perturbation of Minkowski.

Theorem (Marzuola-Metcalfe-T.'07* (stationary))**

Assume that no negative eigenfunctions and zero resonances exist for $\Box_g$. Then local energy decay holds.

A key element here is

Theorem (Kato '59, Agmon '69, ..., Koch-T. '05)

There are no (nonzero) resonances embedded in the continuous spectrum.

Theorem (Sterbenz-T., in progress)

a) (stationary) Bifurcations to negative eigenfunctions for $\Box_g$ can occur only via zero resonances.
b) The result in [MMT] above is stable with respect to small nonstationary perturbations of the metric.

The geometry of black hole space-times

Three distinct regions:

(i) Exterior region $r \gg 1$.
Assumption: asymptotically flat, $g = m + O(r^{-1})$.

(ii) Trapped set $\mathcal{T}$.
Assumptions: (a) hyperbolic trapping (e.g. Zworski-Wunsch),
(b) separate from horizon, and
(c) $\tau \neq 0$ on the trapped set (i.e. ∂_t energy positive there).

(iii) The event horizon $\mathcal{H}$.
Assumption: smooth, nondegenerate red shift and convexity.

Challenges:

- understand the coupling of three regions at high frequency
- the separation between the three regions is blurred at medium and low frequency.

A conditional local energy decay result

Theorem (Sterbenz-T., in progress)

For black hole space-times as above, assume that there are no eigenvalues in $\Im\tau < 0$, and no resonances on $\Im\tau = 0$. Then local energy decay holds. The converse is also true.

A key intermediate step in the above proof is to establish a high frequency local energy decay estimate,

$$(LE) \qquad \|u\|_{LE^1_{\mathcal{T}}} + \|\nabla u\|_{L^\infty L^2} \lesssim \|\Box u\|_{LE^*_{\mathcal{T}}+L^1L^2} + \|\nabla u(0)\|_{L^2} + \|u\|_{L^2_{loc}}.$$

We can also characterize eigenvalues and resonances:

Proposition

a) Eigenvalues and resonances can only occur in a compact subset of $\{\Im\tau \leq 0\}$.
b) Eigenvalues in $\Im\tau < 0$ are smooth and decay exponentially at infinity.
c) Resonances in $\Im\tau = 0$ are smooth and decay like r^{-1} at infinity.

A less conditional local energy decay

Here we make an additional assumption***, that the null generator L extends to a Killing vector field which is time-like near the horizon.

Theorem (Sterbenz-T., in progress)

For black hole space-times as above, assume that there are no eigenvalues in $\Im\tau < 0$, and no zero resonances. Then local energy decay holds.

Ideas:

- Absence of eigenvalues in $\Im\tau < 0 \implies$ subexponential decay.
- The extra assumption above guarantees via Carleman estimates from both infinity and from the horizon, that we have a weaker form of local energy decay for solutions in $[0, T]$, namely

$$(LE) \qquad \|u\|_{LE^1_{\mathcal{T}}} + \|\nabla u\|_{L^\infty L^2} \lesssim \|\Box u\|_{LE^*_{\mathcal{T}} + L^1 L^2} + \|\nabla u(0)\|_{L^2} + \|\nabla u(T)\|_{L^2}.$$

- Coupling the two pieces of information above leads to uniform energy bounds, and thus to local energy decay.

Continuity and stability of local energy decay

Theorem (Sterbenz-T., work in progress)

a) For continuous families of black hole space-times as above, eigenvalues can only bifurcate via a zero resonance.
b) The local energy decay result above is stable with respect to small stationary perturbations.
*c) The local energy decay result above is stable with respect to small nonstationary*** *perturbations.*

*** Some extra condition is needed here near the trapped set.

- One can get local energy decay for Kerr with large a by continuity only by knowing that no zero resonances exist in Kerr.
- The trapped set dynamics are a-priori unstable with respect to small nonstationary nondecaying perturbations.

The Maxwell system

Electromagnetic field F = two form on (M, g).

1. Via differential forms:

$$dF = 0, d * F = 0$$

2. Using covariant differentiation:

$$\nabla^\alpha F_{\alpha\beta} = 0, \qquad \nabla_{[\gamma} F_{\alpha\beta]} = 0$$

3. Using electromagnetic potential A, $F = dA$:

$$\nabla^\alpha \nabla_\alpha A_\beta = 0, \qquad \nabla^\alpha A_\alpha = 0 \quad \text{(gauge condition)}$$

4. Expressed in a reference frame (Neumann-Penrose formalism)

The Maxwell energy

Energy-momentum tensor

$$T_{ij} = g^{kl} F_{ik} F_{lj} + \frac{1}{4} g_{ij} F_{kl} F^{kl}$$

$$\nabla^i T_{ij} = 0$$

If X is Killing then

$$\nabla^i (T_{ij} X^j) = 0$$

and one obtains a conserved energy,

$$E_X(F) = \int_{\Sigma_t} *i_X T = \int_{\Sigma_t} \nu^i T_{ij} X^j dV_\Sigma$$

Positive definite if X is timelike and Σ is space-like. Then

$$E_X(F) \approx \|F\|^2_{L^2(\Sigma_t)}$$

General considerations

- the same three high frequency regions: (i) the exterior region, (ii) the trapped region and (iii) the event horizon, with the same high frequency energy dynamics
- the red shift effect is effective at the level of L^2 solutions for familiar space-times (e.g. Schwarzschild/Kerr)
- additional difficulty at zero frequency arising from charges.
- Modified form of local energy decay, to account for charges.

The low frequencies and charges

For a closed two dimensional surface S define the electric charge inside S by

$$Q = \int_S F$$

Magnetic charge inside S:

$$Q^* = \int_S F^*$$

It is natural to take S which includes the black hole inside. Then these are conserved quantities for the homogeneous problem.
Hodge dual stationary solutions in Schwarzschild:

$$F_0 = \frac{Q}{4\pi} d\omega_{\mathbb{S}^2}, \qquad F_0^* = \frac{Q^*}{4\pi} r^{-2} dr \wedge dt$$

There is a straightforward modification for Kerr.

Local energy decay

Bound for the homogeneous equation:

$$\|F\|_{LE_{\mathcal{T}} \cap L^\infty L^2} \lesssim \|F(0)\|_{L^2}$$

for charge free solutions.

Inhomogeneous equation:

$$dF = G, \qquad dF^* = G^*$$

Modified local energy decay:

$$\|F\|_{LE_{\mathcal{T}}} + \|rF_{rad}\|_{LE} \lesssim \|F(0)\|_{L^2} + \|(G, G^*)\|_{LE^*_{\mathcal{T}}} + \|r(G, G^*)_{rad}\|_{LE^*}$$

Poinwise decay

Price law:

$$|F| \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^3}$$

Peeling estimates (Penrose, Klainerman)

$$|F(\bar{L}, e)| \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^3}$$

$$|F(\bar{L}, L)| + |F(e, e)| \lesssim \frac{1}{\langle r\rangle\langle t\rangle\langle t-r\rangle^2}$$

$$|F(L, e)| \lesssim \frac{1}{\langle r\rangle\langle t\rangle^2\langle t-r\rangle}$$

Here $L = \partial_t + \partial_r$, $L^* = \partial_t - \partial_r$.
Null frame $(L, \bar{L}, e_A, e_B)$.

The results so far

Theorem (Sterbenz-T '13)

Consider a spherically symmetric black hole space-time as above. Then:
a) Uniform energy estimates hold for Maxwell.
b) Local energy decay holds for Maxwell.

Ongoing work: Spectral characterization of local energy decay for nonradial metrics, similar to the scalar case

Theorem ((Price Law) Metcalfe-Tohaneanu-T., almost ready)

Assume that uniform energy estimates and local energy decay hold for Maxwell. Then pointwise decay estimates hold.

This last result does not require the metric to be radial or stationary.