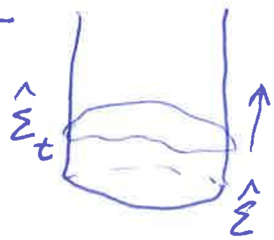
Fig 1

Fig 2



THE PENROSE INEQUALITY FOR ASYMPTOTICALLY LOCALLY HYPERBOLIC SPACES WITH NONPOSITIVE MASS

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(Riemannian) Penrose inequality: If (M^3, g) is a complete asymptotically flat (AF) manifold with boundary, with scalar curvature $R \geq 0$ and with ∂M being an outermost minimal surface ("horizon," i.e. no other minimal surfaces contain it.), then $m \geq \sqrt{\frac{|\partial M|}{16\pi}}$. We call this the (Riemannian) Penrose inequality because this version is only for time symmetric slices. See figure 1. We get equality only for a (Riemannian) slice of Schwarzschild.

It was originally proved by Huisken-Ilmanen using IMCF (inverse mean curvature flow, i.e. $\frac{dx}{dt} = \frac{\nu}{H}$). It was independently proved by Bray using conformal flow. It was generalized to dimensions < 8 by Bray-L using conformal flow. There is a more general version, the "spacetime" Penrose inequality, which is still open.

Today: Discuss an analog of this for asymptotically locally hyperbolic (ALH) spaces.

The Schwarzschild solutions comes from a natural construction; we look for spherically symmetric AF spaces with $R = 0$. We can write $g = V(r)^{-2}dr^2 + r^2 dg_{S^2}$. We then solve some ODE to get $V = \sqrt{1 - \frac{2m}{r}}$ for some parameter m, the mass, to get $R = 0$. If $m > 0$, then $(M = [2m, \infty) \times S^2, g)$ is a complete AF manifold with a horizon boundary. If $m < 0$ we'd be stuck with something incomplete.

Instead, we can look for AH solutions with $R = -6$. In this case, we get $V = \sqrt{r^2 + 1 - \frac{2m}{r}}$. Again, if $m > 0$, then we get $(M = [r_m, \infty) \times S^2, g)$ is a complete AH manifold with a horizon boundary (at $V = 0$). Similar things happen for $m = 0$ or $m < 0$. This solution is called Schwarzschild AdS or Kottler.

If we solve for $R = -6$, but replace (S^2, g_{S^2}) with a surface with constant curvature $\hat{k}$, $(\hat{\Sigma}, \hat{g})$, we get $V = \sqrt{r^2 + \hat{k} - \frac{2m}{r}}$. If $\hat{k} = 0$, we still need $m > 0$ to get a good solution. But if $\hat{k} = -1$, we only need $m > \frac{-1}{3\sqrt{3}}$ in order for $(M = [2m, \infty) \times \hat{\Sigma}, g)$ to be complete with horizon boundary. This is called "generalized Kottler," or "Kottler of genus $\mathfrak{g}$."

These paces are asymptotically locally hyperbolic (ALH), in the sense that it looks like a quotient of true hyperbolic space $\mathbb{H}^3$ near ∞. These ALH spaces can have negative mass but still be interesting and physically relevant. Much like Schwarzschild, these generalized Kottler metrics are vacuum static, meaning that there is a potential V such that $-V^2dt^2 + g$ solves the Einstein equations with cosmological constant. This construction so far works in any dimension, but we're sticking with dimension 3 since our overall proof only works there.

Definition 0.1. (M^3, g) is ALH if there is a surface $(\hat{\Sigma}, \hat{g})$ with constant curvature $\hat{k}$, such that there is a compact K such that $M \setminus K \simeq (1, \infty) \times \hat{\Sigma}$ with a metric

$$g = (\hat{k} + \rho^2)d\rho^2 + \rho^2\hat{g} + \frac{h}{\rho} + O_2(\rho^{-3}).$$

The first two pieces are exactly the hyperbolic metric. The rest are error terms. Here h is a symmetric 2-tensor on $\hat{\Sigma}$ called the mass aspect tensor.

We define $\mu = \frac{3}{4}\mathrm{tr}_{hatg}h$ as the mass aspect function, and $m = \fint_{\hat{\Sigma}} \mu$ is the mass. We use $\bar{m} = \sup_{\hat{\Sigma}} \mu$. We normalize our space to $\hat{k} = 1, 0, -1$. If $\hat{k} = 0$, then $|\hat{\Sigma}| = 4\pi$.

We will use the Huisken-Ilmanen approach using Geroch monotonicity of the Hawking mass, m_H. We construct a weak IMCF with $\Sigma_0 = \partial M$. We then want to show

$$\sqrt{\frac{|\partial M|}{16\pi}} = m_H(\Sigma_0) \le m_H(\Sigma_t) \to m,$$

which gives the proof. The work is making sure the IMCF still has the properties we want.

We will us a straightforward adaptation of the Hawking mass to $R \ge -6$:

$$m_H(\Sigma) : -\sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \mathfrak{g} - \frac{1}{16\pi}\int_\Sigma (H^2 - 4)\right).$$

Need the -4 to get monotonicity. One hopes we get the same kind of set of inequalities,

$$\sqrt{\frac{|\partial M|}{16\pi}}\left(1 - \mathfrak{g} + \frac{|\partial M|}{4\pi}\right) = m_H(\Sigma_0) \le m_H(\Sigma_t) \to m\gamma$$

where $\gamma = \left(\frac{|\hat{\Sigma}|}{4\pi}\right)^{3/2} = [\max\{1, \mathfrak{g} - 1\}]^{3/2}$.

What's the problem with this approach? Neues found an AH example where $\lim_{t\to\infty} m_H(\Sigma_t) > m = m\gamma$. This looks very bad. However, we observed that if the mass aspect is negative, then this inequality reverses, which is exactly what we want.

Theorem 0.2 (Neues-L)**.** *Let (M^3, g) be a complete ALH manifold with boundary and conformal boundary $(\hat{\Sigma}, \hat{g})$ at infinity with genus $\mathfrak{g}$. Assume $\hat{m} \le 0$. If $R \ge -6$, ∂M is a horizon, and ∂M has a compact piece of genus $\mathfrak{g}$. Then $\bar{m} \ge \frac{1}{\gamma}\sqrt{\frac{|\partial M_1|}{16\pi}}\left(1 - \mathfrak{g} + \frac{|\partial M_1|}{4\pi}\right)$ where ∂M_1 is the piece of boundary with the correct genus. We get equality only for generalized Kottler spaces.*

Corollary 0.3. *Under the same hypotheses,* $\bar{m} > 0$ *for* $\mathfrak{g} = 0$ *or* 1 *and* $\hat{m} > \frac{1}{3\sqrt{3}}$ *for* $\mathfrak{g} > 1$*.*

We also get a new proof of a version of the AH PMT.

AF (Riemannian) PMT: 3 known proofs: Schoen-Yau, using minimal surfaces, for $n < 8$. Witten, using spinors, for spin manifolds. Huisken-Ilmanen, using IMCF, for $n = 3$. For AH PMT: Chrucsiel-Herzlich, Wang did the spinor way. Andersson-Cai-Galloway used minimal surfaces to get $\bar{m} \geq 0$. This work is using the IMCF way.

Proof: If there is a minimal surface, we use Cai's result. If not, we run IMCF starting at a point.

Main application: static uniqueness

Theorem 0.4 (Israel, Müller zum Hagen-Robinsen-Seifert, Bunting-Mosseod-ul-Alam)**.** *The only AF vacuum static initial data sets are Schwarzschild or Euclidean.*

There are some related results for static uniqueness of hyperbolic space.

Definition 0.5. (M^3, g, V) is a complete vacuum static ALH initial data set if $-V^2dt^2 + g$ solves Einstein system with cosmological constant $\Lambda = -3$. Also, $\partial M = \{V = 0\}$.

Fact: $\kappa := |\nabla V|$ is constant on components of ∂M, and is called the surface gravity.

For Kottler metrics, with $\hat{k} = -1$, there is a bijection between surface gravities, which are in $(0, \infty)$, and masses, which are in $(\frac{-1}{3\sqrt{3}}, \infty)$.

Theorem 0.6 (Chrusciel-Simon)**.** *Let* (M^3, g, V) *be a vacuum static ALH initial data set. Assume* ∂M *is homeomorphic to the conformal boundary. Let* (M_0, g_0, V_0) *be the Kottler metric with the same surface gravity. If* $m(\kappa) < 0$*, then* $|\partial M| \geq |\partial M_0|$ *and* $\bar{m} \leq m(\kappa)$*.*

These are in opposition to the ALH PMT as we've shown, and so

Corollary 0.7 (Neues-L combined with Chrusciel-Simon)**.** *Let* (M^3, g, V) *be a complete vacuum static ALH initial data set. Assume* ∂M *is homeomorphic to the conformal boundary and* $m(\kappa) < 0$*. Then* (M, g, V) *is a generalized Kottler. This is a uniqueness result.*

Proof of Penrose inequality. The weak IMCF still works fine, but monotonicity requires that $\chi(\Sigma_t)$ does not jump up. This uses Meeks-Simon-Yau. We need to look at the long time behavior. See figure 2. We compactify, with $s = \rho^{-1}$, by setting $\tilde{g} = \rho^{-2}g = s^2 g$. The maximum height is certainly going to zero. We need more work to show

$$m_H(\Sigma_t) \leq (4\pi)^{-3/2} \left(\int_{\tilde{\Sigma}_t} s^{-2} \right)^{1/2} \left(\int_{\tilde{\Sigma}_t} \mu s \right) + o(1) \leq \bar{m}(\cdots)(\cdots) \leq \bar{m} \left(\frac{|\tilde{\Sigma}_t|}{4\pi} \right)^{3/2}$$

The last one is by Hölder's inequality, which would go the wrong way, except that $\bar{m}$ is negative.

$\square$