Fig 1

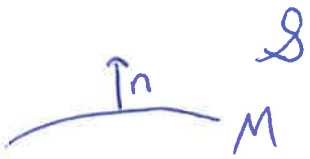


Fig 2

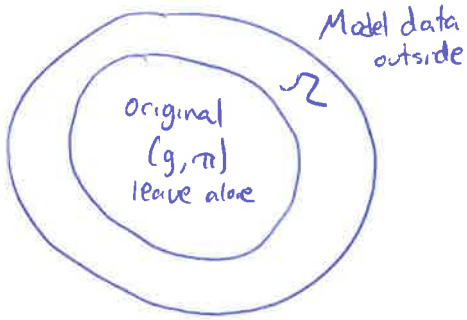
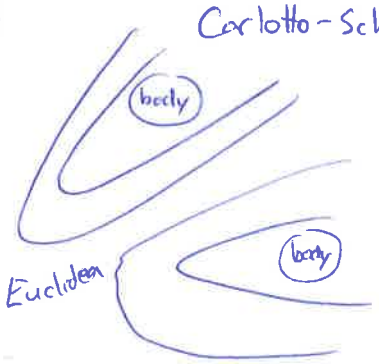
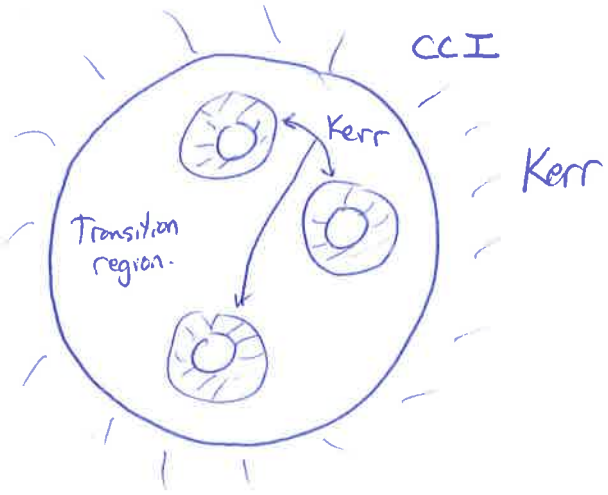


Fig 3





DENSITY AND DEFORMATION THEOREMS FOR THE EINSTEIN CONSTRAINT EQUATIONS

JUSTIN CORVINO

Slide 3: See figure 1.

Slide 9: The energy-momentum functional is continuous in the space $W^{2,p}_{-q} \times W^{1,p}_{-1-q} \ni (g - g_e, \pi)$.

Slide 12: You get this center of mass by taking scalar curvature function, expanding it, then integrating $\int x^l R(g) dx$ by parts. The boundary term is the center of mass.

Slide 13: In the Hamiltonian constraint, $R(g) + Q(\pi)$, the dominant term is $R(g)$.

Slide 21: Cutler-Wald - Using a magnetic field E, they constructed initial Maxwell data (so $R(g) = 2|E|^2_g$), on $\mathbb{R}^3$ such that $|E|$ is compactly supported and rotationally symmetric. For small E, the mass goes to zero. Outside, these were exactly Schwarzschild.

Slide 23: See figure 2.

Slide 24: $\Phi(g + h, \pi + \omega) - (\Phi(g, \pi) + \delta\Phi) \in K_*$.

Slide 31: See figure 3. The bodies may be far away since the cones may be far away.

Why glue Carlotto-Schoen to Schwarzschild outside a large ball? None, really, someone asked me if it was possible.

Density and Deformation Theorems for the Einstein Constraint Equations

Justin Corvino

Lafayette College
U.S.A.

November 22, 2013

Introduction

In this talk, we will discuss first some density theorems for the space of asymptotically flat (AF) solutions to the Einstein constraint equations.

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In this talk, we will discuss first some density theorems for the space of asymptotically flat (AF) solutions to the Einstein constraint equations.

As time permits, we will discuss some applications of localized deformation results, focusing on scalar curvature.

Applications include

(i) a proposition about Bartnik's quasi-local mass,

(ii) scalar curvature and volume, and

(iii) deformations of P-scalar curvature of a metric measure space.

The Constraint Equations

Recall the setup: $M \subset (S, \bar{g})$ is a hypersurface inside an $(n+1)$-dimensional space-time $\mathcal{S}$. The space-time metric satisfies

$$\mathrm{Ric}(\bar{g}) - \tfrac{1}{2} R(\bar{g})\, \bar{g} = T.$$

Initial data

The induced geometric data on M are given by g and π, where g is the induced metric (we assume Riemannian), and π is the momentum tensor: if K is the second fundamental form (with respect to a (local) time-like unit normal vector n^μ, then $\pi = K - (\mathrm{tr}_g K) g$).

Define a scalar function μ by $\mu = T_{\alpha\beta} n^\alpha n^\beta$, and the one-form $\mathcal{J}$, by $\mathcal{J}(X) = -T_{\alpha\beta} n^\alpha X^\beta$, where $X \in T_p M$.

The Constraint Equations

We let $\Phi(g,\pi) = \left(R(g) - \|\pi\|_g^2 + \frac{1}{n-1}(\mathrm{tr}_g(\pi))^2 \ , \ \mathrm{div}_g\pi\right)$ to be the *constraints operator*.

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Vacuum constraints

$\Phi(g,\pi) = (0,0)$.

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(1) Interesting solutions yield interesting space-times.

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For another example, recall that it is a very recent result that on $\mathbb{R}^3$, this moduli space is *connected*, and uses Ricci flow with surgery (F. Codá Marques, Ann. of Math. (2012)).

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(iii) Hybrid (Chruściel-Delay, Chruściel-Isenberg-Pollack, Cortier, Chruściel-Pacard-Pollack, C.-Eichmair-Miao)

Density theorems for the AF constraints

There are LOTS of solutions to the constraints!

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Definition of AF

In an appropriate coordinates x on an asymptotic end (exterior of a ball), $|\partial_x^\beta(g_{ij}(x) - \delta_{ij})| = O(|x|^{-q-|\beta|})$, and $|\partial_x^\gamma \pi_{ij}(x)| = O(|x|^{-1-q-|\gamma|})$, $q \in (\frac{n-2}{2}, n-2]$. We can also formulate this in weighted Sobolev and Hölder spaces.

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In the AF case, the asymptotics are in some sense driven by the ADM energy-momenta, which we will see.

Weighted spaces

(M, g) asymptotically flat. Take $\sigma \geq 1$ a smooth function which in AF coordinates in any end is $\sigma(x) = |x|$.
Define a weighted L^p norm, $p \geq 1$:

$$\|u\|^p_{L^p_{-\tau}} = \int_M (|u|\sigma^\tau)^p \sigma^{-n} \, dv_g.$$

Weighted Sobolev norm

$$\|u\|_{W^{k,p}_{-\tau}} = \sum_{|\gamma| \leq k} \|D^\gamma u\|_{L^p_{-|\gamma|-\tau}}.$$

This gives a Banach space $W^{k,p}_{-\tau}(M, g)$.

Density theorems for AF constraints

Recall the integrals which give the ADM energy, linear momentum, and in case of *enough parity*, the center of mass and angular momentum, of an asymptotically flat solution to the vacuum constraints.

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Energy-linear momentum

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{\{|x|=r\}} \sum_{i,j} (g_{ij,i} - g_{ii,j}) \frac{x^j}{|x|} \, dA_e$$

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{\{|x|=r\}} \sum_{j} \pi_{ij} \frac{x^j}{|x|} \, dA_e.$$

For $|E| \geq |P|$, we define $m = \sqrt{E^2 - |P|^2}$.

Density theorems for AF constraints

Regge-Teitelboim conditions

In an appropriate coordinates x on an asymptotic end,
$|\partial_x^\beta (g_{ij}(x) - g_{ij}(-x))| = O(|x|^{-1-q-|\beta|})$, and
$|\partial_x^\gamma (\pi_{ij}(x) + \pi_{ij}(-x))| = O(|x|^{-2-q-|\gamma|})$,
$q \in (\frac{n-2}{2}, n-2]$. We can also formulate this in weighted Sobolev and Hölder spaces.

Under these conditions the angular momentum and center-of-mass are well-defined. We let Y_i be a basis of rotation Killing fields of Euclidean space, such as $x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$.

Density theorems for AF constraints

Center-of-mass

$$Ec^{\ell} = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{\{|x|=r\}} \sum_{i,j} \Big[x^{\ell}(g_{ij,i} - g_{ii,j}) \frac{x^j}{|x|} - \sum_i \big(g_{i\ell} \frac{x^i}{|x|} - g_{ii} \frac{x^{\ell}}{|x|} \big) \Big] \, dA_e$$

Angular momentum

$$J_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{\{|x|=r\}} \sum_j \pi_{jk} Y_i^k \frac{x^j}{|x|} \, dA_e$$

Density theorems for AF manifolds

Asymptotic integrals

These asymptotic integrals are flux integrals, boundary terms for integration of the constraints operator $\Phi(g, \pi)$ against elements of the kernel of $D\Phi^*_{(g_e,0)}$, where $D\Phi^*_{(g_e,0)}(f, X) = (DR^*_{g_e}(f), L_X g_e)$. The kernel is the direct sum of the span of the constant and linear functions, with the space of Euclidean Killing fields.

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For example, $\mathrm{div}_g \pi = \mathrm{div}_{g_e} \pi + \Gamma * \pi = \mathrm{div}_{g_e} \pi + O(|x|^{-2-2q})$. In case of R-T conditions hold, the error term is even to an extra order.

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So...

Density theorems for AF manifolds

Integrate by parts and use the Killing equation (and symmetry of π), $\mathcal{B}(r)$ is the boundary term, which defines the angular momentum:

$$\int_{r_0\leq|x|\leq r} (\mathrm{div}_g \pi)_k Y_i^k \, dx = \int_{r_0\leq|x|\leq r} (\pi_{jk,j} + O(|x|^{-2-2q}) Y_i^k \, dx$$
$$= \mathcal{B}(r) - \mathcal{B}(r_0) + \int_{r_0\leq|x|\leq r} O(|x|^{-2-2q}) Y_i^k \, dx$$

Without imposing any symmetry, the remaining bulk term is order $|x|^{-1-2q}$, and $\frac{n-2}{2} < q \leq n-2$, so that $3-2n \leq -1-2q < 1-n$.

Integration in spherical gives $|x|^{n-1} \, d|x|$, so that we're integrating between $\int |x|^{2-n} \, d|x|$ and $\int |x|^{\beta} \, d|x|$ for $\beta = -1-2q+(n-1) < 0$.

Density theorems for AF manifolds

Harmonic asymptotics

An initial data set (M, g, π) has *harmonic asymptotics* if outside a compact set (in any given end) in asymptotically flat coordinates we have

$$g_{ij} = u^{4/(n-2)}\delta_{ij}$$
$$\pi_{ij} = u^{2/(n-2)}(X_{i,j} + X_{j,i} - (\mathrm{div}_{g_e}(X))\delta_{ij}).$$

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The energy and linear momenta are encoded in the expansions of u and X:

$$u(x) = 1 + A|x|^{2-n} + O_*(|x|^{1-n})$$
$$X_i(x) = b_i|x|^{2-n} + O_*(|x|^{1-n}).$$

It is easy to show $E = 2A$, and $P_i = -\frac{n-2}{n-1}b_i$.

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It is easy to show $E = 2A$, and $P_i = -\frac{n-2}{n-1}b_i$.
One can also expand u^{odd} and X^{odd} to obtain the center-of-mass and angular momentum parameters, respectively.

Density theorems for AF manifolds

We recall that the constraint equations yield a system of Poisson equations for (u, X), amenable to analysis.

Fundamental density result, roughly formulated

The set of solutions to the constraint equations with harmonic asymptotics is dense in the space of all AF solutions to the constraints, in a suitable topology in which the energy and linear momentum are continuous.

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There are various precise versions of this.

- C.-Schoen: vacuum
- L.-H. Huang: with Regge-Teitelboim conditions
- Huang-Schoen-Wang: specifying angular momentum
- Eichmair-Huang-Lee-Schoen (also C.-Huang, in progress): non-vacuum/dominant energy condition

Density theorems for AF manifolds

Harmonically flat asymptotics (Schoen-Yau)

Suppose (M^3, g) is AF, $R(g) \geq 0$. For any $\epsilon > 0$, there is $\bar{g}$ within ϵ (in $W^{2,p}_{-\delta}$ or $C^{2,\alpha}_{-\delta}$ for $p > 3$, $\frac{1}{2} < \delta < 1$), with $R(\bar{g}) \geq 0$, harmonically flat near infinity in each end, and with $|m(g) - m(\bar{g})| < \epsilon$.

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Remark: H. Bray showed that one can further "bend" the metric (using a clever and elementary superharmonic function argument) so that outside a compact set it is exactly Schwarzschild. Doing so produces some compactly supported positive scalar curvature. Keeping the *vacuum* constraints while making the asymptotic end exactly Schwarzschild seems to take a bit more work.

Density theorem for AF manifolds

Example: the time-symmetric Einstein-Maxwell constraints in three dimensions. The initial data is (M, g, E), with $R(g) = 2|E|_g^2$, $\mathrm{div}_g E = 0$.

Charged harmonic asymptotics

In suitable coordinates in an AF end, $g_{ij} = u^4 \delta_{ij}$ and $E^i = u^{-6}(\mathrm{grad}_{g_e} \phi)^i$, where $u(x) = 1 + \frac{m}{2|x|} - \frac{q^2}{8|x|^2} + O_*(|x|^{-3})$ and $\phi(x) = \frac{q}{|x|} + O_*(|x|^{-2})$.

Used in Khuri-Weinstein-Yamada.

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Used in Khuri-Weinstein-Yamada.

Theorem (C., 2000-2014)

Suppose that (M, g, E) satisfies the dominant energy condition $R(g) \geq 2|E|_g^2$ and is source-free: $\operatorname{div}_g E = 0$. Given $\epsilon > 0$, there is $(\bar{g}, \bar{E})$ on M with $(1 - \epsilon)g \leq \bar{g} \leq (1 + \epsilon)g$, $|E - \bar{E}|_g < \epsilon$, and $|m - \bar{m}| + |q - \bar{q}| < \epsilon$, which outside a compact set on any end solves the constraints $R(\bar{g}) = 2|\bar{E}|_{\bar{g}}^2$, $\operatorname{div}_{\bar{g}} \bar{E} = 0$, and has charged harmonic asymptotics,.

Density theorem for AF manifolds

As another illustration, here is a very interesting density theorem, specifying arbitrary changes in the angular momentum/center of mass.

Theorem (Huang-Schoen-Wang)

Suppose (M^3, g, π) is a nontrivial vacuum initial data set satisfying the Regge-Teitelboim condition. Given any vectors α, $\gamma \in \mathbb{R}^3$, there is a vacuum initial data set $(\bar{g}, \bar{\pi})$ close to (g, π) in $W^{2,p}_{-q} \times W^{1,p}_{-1-q}$, with $\bar{E} = E$, $\bar{P} = P$, $\bar{J} = J + \alpha$, $\bar{C} = C + \gamma$.

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Results in progress (C.-Huang)

Versions of the above with harmonic asymptotics, Regge-Teitelboim condition, and dominant energy condition.

Density theorems for AF manifolds

As another illustrative example, we consider a perturbation to strict dominant energy condition, which is useful in proving the general positive mass theorem.

Theorem (Eichmair-Huang-Lee-Schoen)

Suppose (M, g, π) is AF and satisfies the dominant energy condition. For any $\epsilon > 0$, there is a $\gamma > 0$, and there is an AF initial data set $(\bar{g}, \bar{\pi})$ within ϵ of (g, π) in $W^{2,p}_{-q} \times W^{1,p}_{-1-q}$ so that $\bar{\mu} > (1+\gamma)|\bar{\mathcal{J}}|_{\bar{g}}$.

Density theorems for AF manifolds

As another illustrative example, we consider a perturbation to strict dominant energy condition, which is useful in proving the general positive mass theorem.

Theorem (Eichmair-Huang-Lee-Schoen)

Suppose (M, g, π) is AF and satisfies the dominant energy condition. For any $\epsilon > 0$, there is a $\gamma > 0$, and there is an AF initial data set $(\bar{g}, \bar{\pi})$ within ϵ of (g, π) in $W^{2,p}_{-q} \times W^{1,p}_{-1-q}$ so that $\bar{\mu} > (1+\gamma)|\bar{\mathcal{J}}|_{\bar{g}}$.

The non-strict dominant energy condition can be tough to work with and preserve.

Density theorems for AF manifolds

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A physics motivation

There was interest in constructing such data in hopes the evolution would produce a non-trivial example of an asymptotically simple vacuum (purely radiative) space-time, in the spirit of Cutler-Wald for Einstein-Maxwell (as cited earlier: Chruściel-Delay, 2002; C., 2007; based on work of Friedrich, or now Anderson-Chruściel).

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Instead of just an existence result, we obtained a density theorem.

Density theorems for AF manifolds

Theorem (C.)

Let (M, g) be asymptotically flat, with $R(g) = 0$ (complete, with positive mass, or just an AF end with non-zero mass, say). Given any compact set $K \subset M$, and given any $\epsilon > 0$, there is a metric $\bar{g}$ within ϵ of g (weighted norm, or quasi-isometric, say), with $R(\bar{g}) = 0$, so that in K, $\bar{g} = g$, and in AF coordinates near infinity in any end, $\bar{g}$ has the form $\bar{g}_{ij} = \left(1 + \frac{\bar{m}}{2|x-\bar{c}|^{n-2}}\right)^{4/(n-2)} \delta_{ij}$, for appropriate $\bar{m} \in \mathbb{R}$ and $\bar{c} \in \mathbb{R}^n$, with $|m - \bar{m}| < \epsilon$.

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Generalized to full vacuum constraints by C.-Schoen, Chruściel-Delay; Einstein-Maxwell (C., in progress, still, alas...2014!) The asymptotic model family in these cases is Kerr, or Kerr-Newman, or any *admissible family*.

NOTE: the data is preserved inside a (large) compact region.

Density theorems for AF manifolds

Technique for construction: *controlled localized deformation result* for the constraints operator. Given $\delta\Phi = (2\delta\mu, \delta\mathcal{J})$ small and compactly supported inside a domain Ω, we want to solve for (h, ω) supported in $\overline{\Omega}$ so that $\Phi(g + h, \pi + \omega) = \Phi(g, \pi) + \delta\Phi$.

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Basic estimate

The key ingredient, as R. Schoen mentioned in his talk, is an injectivity estimate of the form $\|(f, X)\| \leq C\|D\Phi^*_{(g,\pi)}(f, X)\|'$ in suitable (weighted) spaces.

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The above holds in case $D\Phi^*_{(g,\pi)}$ has trivial kernel in these spaces, such as in the case R. Schoen discussed (joint work with A. Carlotto).

The estimate is used to solve the linearized problem $D\Phi_{(g,\pi)}(h_0, \omega_0) = \delta\Phi$ variationally.

Density theorems for AF manifolds

Basic estimate

$\|(f, X)\| \leq C \|D\Phi^*_{(g,\pi)}(f, X)\|'$.

In approximation arguments, we want a uniform C for a family of (g, π). If the elements in the family approach a *stationary* initial data set on the domain of interest, so that the limiting operator admits kernel, then the C above cannot be uniform, but can only be uniform when working *transverse* to this limiting (co)-kernel.

Projected problem

In the case with non-trivial kernel, there is a finite-dimensional subspace K_* (kernel elements suitably cut off, say), so that it is possible to solve, with compact support, $\Phi(g + h, \pi + \omega) - (\Phi(g, \pi) + \delta\Phi) \in K_*$.

One can measure how far one is from solving the full problem by projecting this difference into K in $L^2(\Omega)$, say.

Density theorems for AF manifolds

Approximate (co)-kernel

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Density theorems for AF manifolds

Approximate (co)-kernel

For the density theorems via gluing, the approximate solutions produced by gluing approach flat/Minkowski initial data, and so there is a kernel, of dimension ten in case of the three-dimensional constraint equations.

The kernel K corresponds precisely to...the energy, linear momentum, angular momentum and center of mass! (Recall what we saw earlier about projecting the constraints operator into the (co)-kernel.)

We remark that in the work of Li and Yu, an approximate family is constructed which approaches a Schwarzschild metric, where the cokernel is four-dimensional, corresponding the mass and angular momentum (static potential, and rotational symmetries).

Density theorems for AF manifolds

In other words: the PDE methods allow us to solve the constraints up to a finite-dimensional piece. This finite dimensional piece is of the form, where the gluing and compactly supported perturbation is on $r \leq |x| \leq 2r$, of the form

$$\mathcal{B}(2r) - \mathcal{B}(r) + \text{error}.$$

At $|x| = 2r$ is the asymptotic model (Schwarzschild or Kerr), and at $|x| = r$ is the original data...or the other way around if you like! Anyway...

End game

By doing these integrals for all dimensions of the co-kernel, we see that by varying the parameters energy, momentum, angular momentum, center of mass, we can make all the integral vanish—i.e. solve the vacuum constraints!

Note that E and $|P|$ will be close to that of the given initial data, so that $E > |P|$ will be preserved in the construction.

Density theorems for AF manifolds

Admissible asymptotic family

An admissible family is one for which we can effectively vary the parameters to make the above construction work.

Example: Kerr (Chruściel-Delay).

Expected examples of admissible families with harmonic asymptotics (C.-Huang).

Density theorems for AF manifolds

From what we described above, it would seem natural that we would, for the purposes of gluing an asymptotic end onto a given initial data set, assume R-T asymptotics to start, so that the center-of-mass and angular momentum are well-defined.

It turns out you don't need to do this! The upshot is that (as in other density theorems, cf. e.g. Huang, C.-Huang) when R-T is assumed, the density theorem can be carried out in a topology respecting the R-T condition, and the resulting configuration has center-of-mass and angular momentum *close* to that of the original. But natural density theorems still exist in spaces where the energy-momentum is well-defined.

Theorem/Observation (Chruściel-C.-Isenberg)

The set of vacuum initial data with Kerr ends is dense in the space of AF vacuum initial data, *without* assuming the R-T condition.

Density theorems for AF manifolds

Let's briefly discuss the mechanism for this, and then make a couple final remarks.

The gluing happens in an annulus $A_r = \{r \leq |x| \leq 2r\}$, say. Let $\phi_r : A_1 \to A_r$ be $\phi^r(x) = rx$. Let $g^r = r^{-2}\phi_r^* g$ and $\pi^r = r^{-1}\phi_r^*\pi$.

$$r \int_{|x|=1} \pi^r_{jk} Y_i^k \frac{x^j}{|x|}\, dA_e = r^{-1} \int_{|x|=r} \pi_{jk} Y_i^k \frac{x^j}{|x|}\, dA_e$$

which as we saw earlier is (from the bulk term estimate) $r^{-1}\mathcal{B}(R_0) + \text{error}$, where the error term is $O(r^\beta)$ for $\beta < 0$, or $O(\frac{\log r}{r})$, in either case $o(1)$.

In the R-T case, get better estimate $O(r^{-1})$.

Density theorems for AF manifolds

This translates after re-scaling as follows: while $|\Delta E|$ (the change in ADM energy from the initial to the exterior Kerr value) is small, say $O(r^{-1})$, the parameter values of ΔJ and Δc for the exterior Kerr might be reasonably large.

Of course, if the initial data is not R-T, then it may not have possessed an initial J and c to start, so this is reasonable.

Theorem (C.-Huang)

The space of vacuum AF data with infinite J and c is $W^{2,p}_{-q} \times W^{1,p}_{-1-q}$ dense in the space of vacuum data.

Density theorems for AF manifolds

We make some concluding remarks on N-body configurations.

R. Schoen described his work with A. Carlotto, constructing N-body initial data sets, in which each body is placed inside a cone region, and various cone regions can be placed into a single AF end with *flat* data in between.

CCI construction

The N-body configurations of Chruściel-C.-Isenberg are constructed differently: we construct template regions which are Kerrian near N-points (think Kerr in annuli around N points) of various parameter values. We glue these to a master exterior Kerr, and fine tune the parameters (as discussed above) to solve the vacuum constraints, leaving Kerr solutions around each point, and near infinity. Given N bodies, then, we can first glue each body into a Kerr exterior, and then insert into the template.

Density theorems for AF manifolds

Remark: Using the CCI observation—don't need RT or harmonic asymptotics to glue at infinity—it appears that one could glue the Carlotto-Schoen data to a Schwarzschild exterior region, of total mass roughly the sum of the masses of the N bodies, near infinity.

The procedure must smear out the part of the metric carrying the mass from the transition regions between the cones.

This gives examples (there are others) with a flat region, and a Schwarzschild exterior, with zero scalar curvature.

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There are LOTS of solution of the constraints!