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Fig 1:

$$
\begin{tikzcd}
\mathrm{SH}\ \mathbb{T}^3\text{-Gowdy} \arrow[r, hook, "\text{unstable}"] \arrow[d, hook] & \text{general } \mathbb{T}^3\text{-Gowdy} \arrow[d, hook, "\text{unstable}"] \\
\mathrm{PH}\ \text{general } \mathbb{T}^2 \arrow[r, hook, "\text{unstable}"'] & \text{general } \mathbb{T}^2
\end{tikzcd}
$$

STABILITY AND INSTABILITY OF SPATIALLY HOMOGENEOUS SOLUTIONS IN THE $\mathbb{T}^2$ SYMMETRIC SETTING

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I'm worried about future asymptotics of $\mathbb{T}^2$ spacetimes.

Personal motivation: A general motivation is that in cosmology, we use spatially homogeneous spaces to model cosmology. (Here will mostly use vacuum without cosmological constant). A more specific motivation is this conjecture: Let (M, g) be vacuum with constant mean curvature (CMC) spatially compact slices Σ_τ. We say $\tau \to 0^-$ is the expanding direction. We assume the future is future geodesically complete. On these slices, we get metrics $\hat{g}_\tau$ on Σ_τ. We can think of these as defined on some Σ since all the slices are diffeomorphic. It is then natural to rescale the metrics to see behavior; we call these rescaled metrics g_τ, which are on Σ. To do this, we take proper time squared and multiply it by the metric. We decompose $\Sigma = G \cup H$ where G is graph manifold pieces, and H is hyperbolic pieces. We expect that g_τ converges on H (complete, finite volume), but collapses on G. We should get isotropization on the H pieces. Thus the average observer sees something isotropic. Everything known fits into this.

Model case: Milne model: $-dt^2 + t^2 g_H$. We get the desired picture. If you divide by t^2, you see the convergence we expect. This behavior is stable/ is an attractor. Despite this model, we don't have a large class of solutions which have both the convergence and collapse behavior. What would be the model case for the graph part? [Answer: Lots of things we might expect don't work because they're unstable.]

Q: Is there a model solution in the graph manifold setting? Kasner solutions collapse and are graph manifolds, but they are not stable, and so aren't good models. This leads us to consider stability and instability.

We consider a metric in areal coordinates:

$$g = t^{-1/2}e^{\lambda/2}(-dt^2 + \alpha^{-1}d\theta^2) + te^P[dx + Qdy + (G + QH)d\theta]^2 + te^{-P}(dy + Hd\theta)^2$$

on $(t_0, \infty) \times \mathbb{T}^3$ and $t_0 \geq 0$.

Thus we get polarized solutions if $Q = 0$. We get $\mathbb{T}^3$-Gowdy solutions if $G = H = 0$.

We next consider the Einstein equations in these coordinates.

(1)

$$P_{tt} + \frac{1}{t}P_t - \alpha P_{\theta\theta} = \frac{\alpha}{2}P_\theta + \frac{\alpha_t}{2\alpha}P_t e^{2P}(Q_t^2 - \alpha Q_\theta^2) - \frac{e^{p+\lambda/2}k^2}{2t^{7/2}}$$

(2)
$$Q_{tt} + \frac{1}{t}Q_t - \alpha Q_{\theta\theta} = \frac{\alpha_\theta}{2}Q_\theta + \frac{\alpha_t}{2\alpha}Q_t - 2(Q_tP_t - \alpha Q_\theta P_\theta)$$
(3)
$$\frac{\alpha_t}{\alpha} = -\frac{e^{P+\lambda/2}k^2}{t^{5/2}}$$
(4)
$$\lambda_t = t[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] - \frac{e^{P+\lambda/2}k^2}{t^{5/2}}$$
(5)
$$\lambda_\theta = 2t(P_tP_\theta + e^{2p}Q_tQ_\theta)$$
[I'm fairly certain these equations are correct, but it was fast. There may be mistakes.] The 5th equation is a constraint equation. If $k = 0$, we get a $\mathbb{T}^3$-Gowdy solution. This assumption makes the equations easier since they decouple.

Pseudo-homogeneity: We say a solution (P, Q, λ) is pseudo-homogeneous if they are independent of θ. There is then a natural energy,
$$\hat{H} = \int_{S^1}\left(t^2\alpha^{-1/2}\left[P_t^2 + \alpha P_\theta^2 + e^{2P}\left(Q_t^2 + \alpha Q_\theta^2\right)\right] + 3\alpha^{-1/2} + \frac{\alpha^{-1/2}e^{P+\lambda/2}k^2}{t^{3/2}}\right)d\theta.$$
For this energy, $\partial_t\hat{H} \geq 0$ but $\partial_t(t^{-2}\hat{H}) \leq 0$.

Some people have shown that solutions are inextendible to the future. Berger et al showed that solutions are future global.

Results: Polarized, spatially homogeneous $\mathbb{T}^3$-Gowdy: In this case, the equations become $\partial_t(tP_t) = 0$ and $\lambda_t = tP_t^2$. Thus $P(t) = r_\infty \ln t + C_P$ and $\lambda_t = r_\infty^2 \ln t + C_p$. This includes all Kasner solutions. Are these solutions stable?

If we add a spatial variation we get
$$P_tt + \frac{1}{t}P_t - P_{\theta\theta} = 0$$
$$\lambda_t = t(P_t^2 + P_\theta^2)$$
$$\lambda_\theta = 2tP_tP_\theta$$
Then these equations have solutions $P(t, \theta) = r_\infty \ln t + c_p + t^{-1/2}\nu(t, \theta) + \psi(t, \theta)$ where $v_tt = v_{\theta\theta} = 0$, $\int_{S^1}\nu(\cdot, \theta)d\theta = 0$ and $\psi = O(t^{-3/2})$ and $\int_{S^1}\psi(\cdot, \theta)d\theta = 0$. There is a unique solution with this data at infinity.

The leading order is spatially homogeneous, but it's not okay if we're looking at λ. If ν is not identically zero, then λ goes to infinity linearly. Thus all the expansion is in one direction. (instead of 2 directions and 1 contracting as normal for Kasner).

Polarized $\mathbb{T}^2$: Pseudo-homogeneous solutions with $k \neq 0$: there exists a unique (pseudo-homogeneous) solution with the asymptotics in this paragraph. We take constants $c_P, c_\lambda, r_\infty \in (-3, 1)$ and $\alpha_\infty \in C^\infty(S^1, \mathbb{R}_t)$. We then have the properties $\alpha(t, \theta) \to \alpha_\infty(\theta)$, $P_t - r_\infty \ln t - c_p \to 0$, $\lambda(t) - r_\infty^2 \ln t - c_\lambda \to 0$.

Proposition 0.1. *Take $(P_{bg}, \lambda_{bg}, \alpha_{bg})$, a pseudo-homogeneous solution on $(t_0, \infty) \times S^1$, $K \neq 0$. Let $t_a \in (t_0, \infty)$. Then there is an $\epsilon > 0$ such that if (P, Q, λ) is a non-pseudo-homogeneous solution, such that*

$$\|P - P_{bg}(t_a, \cdot)\|_{C^1} + \|\delta_t(P - P_{bg})(t_a, \cdot)\|_{C^0} + \|(\alpha - \alpha_{bg})(t_a, \cdot)\|_{C^1} + \|(\lambda - \lambda_{bg})(t_a, \cdot)\|_{C^1} \leq \epsilon.$$

Then

$$\lim_{t\to\infty} \|\alpha(t, \cdot)\|_{C^0} = 0$$
$$\lim_{t\to\infty} \|P(t, \cdot)/\ln t + 1\|_{C^0} = 0$$
$$\lim_{k\to\infty} \|\lambda(t_k, \cdot)/\ln t_k - 5\|_{C^0} = 0.$$

This is compared to

$$\alpha_{bg}(t, 0) \to \alpha_\infty(\theta) > 0$$
$$P_{bg}(t)/\ln t \to r_\infty$$
$$\lambda_{bg}(t)/\ln t \to r_\infty^2$$

for pseudo-homogeneous solutions.

Thus, the slightest bit of spatial variation gives something very different.

Theorem 0.2. *Consider a solution to* (1) − (5) *with $K \neq 0$ (i.e. not Gowdy). If $< \alpha^{-1/2} >$ (the mean value over theta) is bounded, the solution is pseudo-homogeneous.*

This quantity is increasing, so it is either bounded or goes to infinity. Since perturbations change the asymptotics of this, pseudo-homogeneity is unstable. In short, spatially homogeneous $\mathbb{T}^3$-Gowdy is sitting unstably in general $\mathbb{T}^3$-Gowdy. Etc. See figure 1. Thus there is no hope really, of these being good, stable, model for graph manifolds.

Outline of proof of Theorem:

There is a nice energy which is increasing, as before. Let $g = P + \frac{1}{2}\lambda - \frac{1}{2}\ln\alpha$, $f = \alpha^{-1/2}e^{P+\lambda/2}$. Then

$$\begin{aligned} e^{<g>} &\leq < e^g > \\ &\leq < f > \\ &\leq \frac{1}{2\pi}k^{-2}t^{3/2}\hat{H} \\ &\leq Ct^{7/2}. \end{aligned}$$

Thus $< g > \leq \frac{7}{2}\ln t + C$. And so

$$\frac{1}{2\pi}\int_{t_1}^{t}\int_{S^1} g_t d\theta dt \leq \frac{7}{2}\ln t + C$$

where $t \geq t_1 := t_0 + 2$. Then

$$\int_{t_1}^{t} \int_{S^1} s[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] d\theta ds \leq C \ln t$$

for $t \geq t_1$. A $\mathbb{T}^3$-Gowdy spacetime that satisfies this must be spatially homogeneous. Also, the mean of λ can't grow faster than $\sqrt{t}$, unlike normally, where it grows linearly. This is a general estimate that holds for any $\mathbb{T}^2$ metric.

If there exists an $\alpha_0 > 0$ such that $\alpha(t,\theta) \geq \alpha_0$ for any $(t,\theta) \in [t,\infty) \times S^1$. Then $\int_{t_1}^{t} \frac{1}{s} \hat{H}(s) ds \leq C \ln t$ for $t \geq t_1$. Since $\hat{H}$ is increasing, we have $\hat{H} \leq C$. This tells you that L^2 norm of [something] is bounded? and thus $\| P - < P > \|_{C^0} \leq C t^{-1}$.

Step 2: Prove that the solution has pseudo-homogeneous asymptotics. This is long and hard, but let's just assume this. Then exists a uniquely associated pseudo-homogeneous solution. We want to prove that these are indeed the same.

Let's explain in a simple special case, polarized $\mathbb{T}^3$-Gowdy. Let's assume

$$\lim_{t \to \infty} t^2 \int_{S^1} [(P_t - \partial_t P_{hom})^2 + P_\theta^2] d\theta = 0.$$

Recall, $t \partial t P_{hom} =$ [missed this]. If we let

$$\hat{H}_G = \int_{S^1} (P_t^2 + P_\theta^2) d\theta,$$

we get

$$\frac{d\hat{H}_g}{dt} = 2t \int_{S^1} P_\theta^2 d\theta.$$

Thus

$$\hat{H}_G(t) \leq \lim_{t \to \infty} \hat{H}_G(t) = \hat{H}_{G,hom},$$

where $\hat{H}_G$ is increasing to that.

Moreover, $A = \int_{S^1} t P_t d\theta$ is conserved. By assumption, it must be the same for both solutions. Thus

$$\begin{aligned}
2\pi \hat{H}_{G,hom} &= A^2 \\
&= t^2 \left(\int_{S^1} P_t d\theta \right)^2 \\
&\leq 2\pi t^2 \int_{S^1} P_t^2 d\theta \\
&= 2\pi \hat{H}_G(t) - 2\pi t^2 \int_{S^1} P_\theta^2 d\theta \\
&\leq 2\pi \hat{H}_{g,hom} - 2\pi t^2 \int_{S^1} P_\theta^2 d\theta.
\end{aligned}$$

Thus $2\pi t^2 \int_{S^1} P_\theta^2 d\theta \leq 0$, and thus the spatial variation had to be zero to start.

In the general case, life is not this nice. We can construct an energy between the two solutions. We can prove a lower bound on the decay rate. Also, there are conserved quantities, which we can use to get a lower bound on the kinetic part of the energy. We then use monotonicity to get another bound. It follows the same general type of idea. We write down a system of estimates, and can iteratively improve them. We get a decay rate that is faster than the lower bound from the energy estimates, and so the solutions must be the same.

On the other hand, positive cosmological constant very easily gives stability.