Fig 1

Fig 2

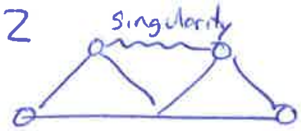

Fig 3

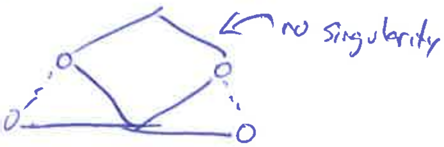

Fig 4

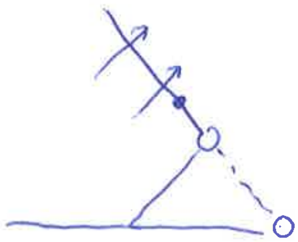

Fig 5

Fig 6

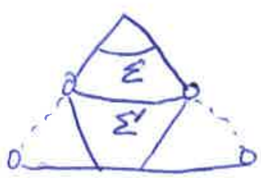

Fig 7

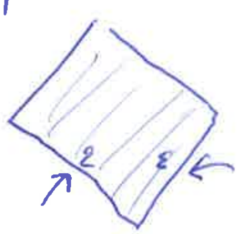

Fig 8

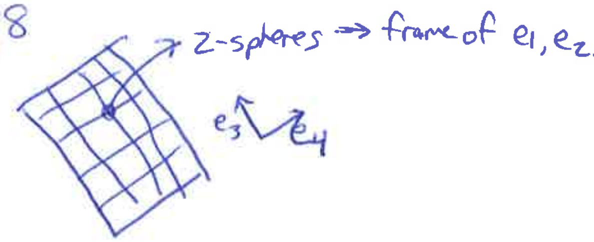

Fig 9

WEAK NULL SINGULARITIES IN GENERAL RELATIVITY

JONATHAN LUK

Take (M^4, g) with signature $(-,+,+,+)$. I'll mostly be worried about $\text{Ric}(g) = 0$, the vacuum case.

Ex: Minkowski: See fig 1. Schwarzschild: See fig 2. Kerr: See fig 3. For Kerr, there's a problem of determinism: there exists non-unique smooth extensions to Kerr which are solutions to the vacuum Einstein equations. You can evolve off of the smooth Cauchy horizon using a characteristic development.

Strong cosmic censorship conjecture: the maximal Cauchy development (MCD) to generic asymptotically flat (AF) initial data is future inextendible as a sufficiently regular Lorentzian manifold.

Minkowski can't be extended. Schwarzschild can't be extended with a C^2 metric. Kerr can be extended. According to this conjecture then, Kerr is non-generic. Thus we expect the Kerr extension to be unstable.

Why do we expect this? What is the instability mechanism for the Cauchy horizon? Blue shift. Look at figure 4. If two observers are crossing the horizon, and one sends a signal to the other, it will be blue shifted when they receive it. We can consider a solution to $\Box\phi = 0$ in the interior of a black hole. (We do this for Reissner-Nordstrom.)

Proposition 0.1. *Suppose $|\partial_v \phi| \le C v^{-p}$ for some large p and the Eddington-Finklestein v (this is data on horizon). In the whole region inside the black hole in Fig 4, the solution is bounded. If we look at a regular coordinate system at a point on the Cauchy horizon, $|\partial_V \phi| < \frac{c}{V \log^p(c/V)}$.*

There exists spherically symmetric data such that in a regular coordinate system, $|\partial_V \phi| \ge \frac{C'}{V \log^p(c/V)}$. In particular, $\partial_V \phi \in L^1 \setminus L^q$ for any $q > 1$.

Theorem 0.2 (Dafermos)**.** *A. Consider the spherically symmetric Einstein-Maxwell-Scalar field (SSEMSF) model with data on the horizon as above.*

1. *There exists a solution in a region as in Figure 5.*
2. *The solution extends continuously to the boundary.*
3. *The scalar field obeys bounds as above.*
4. *There exists data such that Christoffel symbols are not in L^2.*

This means that you cannot extend the spacetime, since the Einstein equations don't make sense if the Christoffel symbols are not in L^2.

"Definition" of Weak null singularities - null singularities, continuous up to boundary, and Christoffel symbols are not in L^2.

Theorem 0.3 (Dafermos,Dafermos-Rodnianski)**.** *B. Suppose we have globally small perturbations to Reisner-Nordstrom for the SSEMSF system. Then you have the Penrose diagram as in figure 6. Also, the previous theorem holds for these spaces.*

Question: Are the Dafermos spacetimes stable (outside spherical symmetry)?

Theorem 0.4 (L)**.** *1. Take a surface Σ as in figure 6. For small perturbations of the Dafermos spacetimes on Σ (outside spherical symmetry), there exists a solution in the region above Σ where the metric and scalar field are continuous up to boundary, and if the background is singular, then the perturbed spacetime is also singular.*

Could you perturb it more globally?

Theorem 0.5 (Dafermos-L)**.** *2. The same holds for Σ'. This surface is of constant r (since it comes for a spherically symmetric space, this is well defined).*

Remark 1: Theorem 1 follows from a more general local existence result: See figure 7: using characteristic initial formulation, taking $|\partial_v\phi| < \frac{C}{v\log^p(c/v)}$ and $\chi \leq \frac{c}{v\log^p(c/v)}$ for $p > 1$ on the right side. Similarly for the left side, but for $\underline{\chi}$. All analogous derivatives satisfy the same bounds. Then there exists ϵ such that the development shown exists.

Remark 2: We can consider all these in vacuum. For the global theorem, one can have the analogous statement for the constant r surface Σ' in Kerr.

Remark 3: Conjecture: Theorem A and B hold in vacuum (with the Reissner-Nordstrom spacetime replaced with Kerr.)

Ideas from proof: 1. Renormalized energy estimates: comes from impulsive gravitational waves - joint with Rodnianski. The curvature tensor is not in L^2. We use a double null foliation. See figure 8. We decompose curvature and Ricci coefficients with respect to this frame. None of the null decomposed curvature components are in L^2.

- $\alpha \sim \log^p(1/v)/f(v)^2$
- $\beta \sim 1/v\log^p(1/v) := 1/f(v)$
- $\rho, \sigma \sim 1/f(u)f(v)$
- $\underline{\beta} \sim 1/u\log^p(1/u) := 1/f(u)$
- $\underline{\alpha} \sim \log^p(1/u)/f(u)^2$

If this is the case, then everything is too singular. But then one can find renormalizations of ρ and σ which behave better. We estimate K, the Gauss curvature, $K = \rho + \Gamma\Gamma$, quadratic in the Ricci coefficients, or $= \sigma + \Gamma\Gamma$ and these don't see singularities.

We first write down estimates for β, $\underline{\beta}K$ and $\sigma + \Gamma\Gamma$. We find these estimates decouple from that of α and $\underline{\alpha}$.

2. Weighted estimate: We incorporate the singularities into the norms: $\int f(v)\beta^2 dv$. This overall acts like $1/f$, which is integrable, so the integral is finite. Integration

by parts gives very singular terms though, which is bad. The weights introduce singular terms not controllable by flux term. But these singular terms have a good sign.

3. Null structure of the Einstein equations: Can use it to get estimates.

Idea for proof of the global problem: We still need a double null foliation, and then we renormalize. We have to consider the difference with the background solution. The incorporated weights capture the blue shift from the background. We then use an integrated decay estimate on the shaded area in figure 9. This degenerates towards the Cauchy horizon. We want to find a hypersurface γ such that its future has finite volume (as in figure 9). Before this surface, the decay estimate is strong enough to get a whole solution below it, and then we can use the local result for the rest.

Do you impose strong restrictions on the data near the edges of Σ'? We need polynomial decay on the data.