Lan-Hsuan Huang

Fig 1

Fig 2

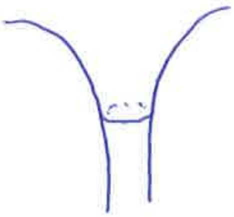
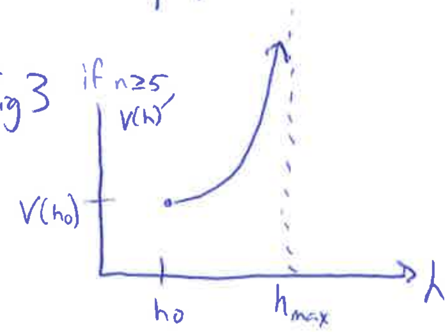


Fig 3



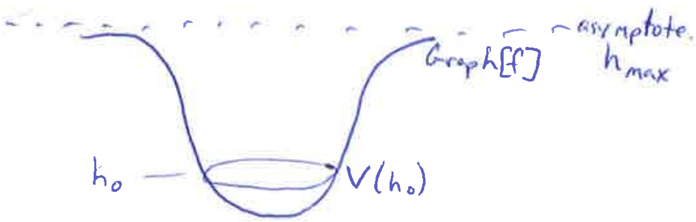


Fig 4

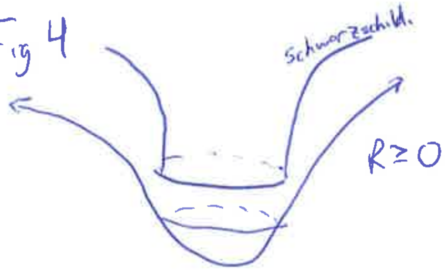
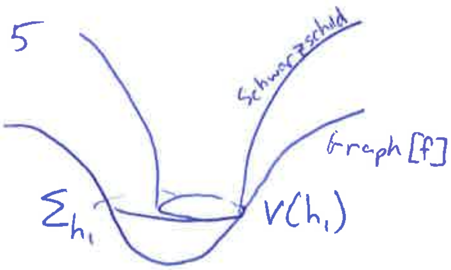


Fig 5



ASYMPTOTICALLY FLAT GRAPHS WITH SMALL MASS

LAN-HSUAN HUANG

We define an asymptotically flat graph as follows: take $f : \mathbb{R}^n \to \mathbb{R}$ with $|Df| \to 0$ and $f \to c$ or ∞ as $|x| \to \infty$. Then Graph$[f] \subset \mathbb{R}^{n+1}$ has the induced metric $g_{ij} = \delta_{ij} + f_i f_j$.

Reily: The scalar curvature of this is $R = \partial_i \left(\frac{f_{ii} f_j - f_{ij} f_j}{1+|Df|^2} \right)$.

Lam: By the divergence theorem (see fig 1), we get $c(n)m = \int_{R^n} R\, dx$, the ADM mass, where $c(n) = 2(n-1)\omega_{n-1}$. This immediately gives the positive mass theorem (PMT).

We also get the Penrose inequality: Let $\Sigma = \partial\Omega := \{f^{-1}(h)\}$. Thus

$$c(n)m = \int_{R^n \setminus \Omega} R\, dx + \int_{\Sigma} \frac{|Df|^2}{1+|Df|^2} H_\Sigma$$

where H_Σ is the mean curvature of Σ in $\mathbb{R}^n \times \{h\}$.

Examples of AF graphs: Schwarzschild of mass m. Outside the minimal surface, we have $f(x) = \sqrt{\delta m(|x| - m)}$ for $n = 3$, $= \sqrt{2}m(\ln|x| + \sqrt{|x^2 - 2m})$ for $n = 4$ and is $O(|x|^{2-\frac{n}{2}})$ for $n \geq 5$.

What about the rigidity case? H-Wu: H_Σ has a sign. If $\vec{H}$ is the mean curvature of the graph in $\mathbb{R}^{n+1}$, then $\langle \vec{H} \vec{H}_\Sigma \rangle \geq \frac{R}{2}$. Also, $H \geq 0$. Thus H_Σ has a sign. This gives the rigidity case of PMT, since both terms are nonnegative, and thus $H_\Sigma \equiv 0$. This contradicts the boundedness of Σ, which implies that $\Sigma = \emptyset$ which implies $f \equiv c$.

What about stability for PMT? If the mass is small, what can you say about the graphing function? We want that the graph is close to a plane.

Finster, Bray-Kath used a spinor argument to get a bound on the L^2-norm of curvature except a set of small measure. (These are for general AF manifolds, not just graphs.)

Corvino: Assume a uniform bound on the sectional curvature in addition to small mass. Then small mass implies that the AF manifold is diffeomorphic to $\mathbb{R}^n$.

Lee: Convergence as the mass goes to zero can't be in a strong sense, since we could have a long, thin neck after/below a minimal surface, as in fig 2. Assume the AF manifold is conformally flat and scalar flat (outside a compact set). Then the manifold is close to the Euclidean metric outside a compact set.

What about inside a compact set? Lee-Sormani: Suppose the AF manifold is rotationally symmetric. Then small mass implies a compact set is close to Euclidean space in the intrinsic flat distance. We also have to assume that there are no minimal surfaces inside the compact set. This actually means that it can be embedded as a graph.

Joint work with Lee: What can we say about an AF graph with small mass? When mass is small, we see that the total scalar curvature is small, and the integral over the level set is small. The mass will give a bound for $\int_\Sigma \frac{|Df|^2}{1+|Df|^2} H_\Sigma \leq c(n)m$. If we didn't have the $|Df|$ term, there are elementary inequalities for this.

We have

$$c(n)m \geq \int_\Sigma \frac{|Df|^2}{1+|Df|^2} H_\Sigma \geq \frac{\alpha^2}{1+\alpha^2} \int_{\Sigma \cap \{|Df| \geq \alpha\}} H_\Sigma.$$

Define $V(h) = H^{n-1}(\Sigma_h)$ (Hausdorff measure) where $\Sigma_h = \{f^{-1}(h)\}$. Then

$$V'(h) = \int_{\Sigma_h} \frac{1}{|Df|} H_{\Sigma_h} > \frac{1}{\alpha} \int_{\Sigma_h \cap \{|Df| < \alpha\}} = \frac{1}{\alpha} \left(\int_{\Sigma_h} H_\Sigma - \int_{\Sigma_h \cap \{|Df| < \alpha\}} H_\Sigma \right).$$

Minkowski integral formula: Assume $H_\Sigma > 0$ (we already had $H_\Sigma \geq 0$) and either Σ is outer minimizing (Huisken) or Star-shaped (Guan-Li). Then

$$\int_\Sigma H_\Sigma \geq \frac{c(n)}{2} \left(\frac{V(h)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Thus

$$V'(h) > \frac{c(n)}{\alpha} \left(\frac{1}{2} \left(\frac{V(h)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} - (1+\alpha^{-2})m \right).$$

This inequality is not optimal. For Schwarzschild graph, if we take

$$\alpha_0 = |Df| = \left(\frac{1}{2m} \left(\frac{V(h)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} - 1\right)^{-1/2}$$

and plug it in, we get the trivial inequality $V' \geq 0$. The optimal inequality is given by

$$\alpha = \sqrt{3} \left[\frac{1}{2m} \left(\frac{V(h)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} - 1 \right]^{-1/2}.$$

If we plug in this α, we get

$$V'(h) > c(n) \frac{2m}{3\sqrt{3}} \left[\frac{1}{2m} \left(\frac{V(h)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} - 1 \right]^{3/2}$$

Thus the leading power is $\frac{n-2}{n-1}\frac{3}{2} > 1$ if $n \geq 5$. Thus, the area will blow up in finite time. Thus we have a maximum height of f since $V(h)$ must blow up in finite time. This inequality is valid if $h \geq h_0$ where

$$\frac{1}{2m} \left(\frac{V(h_0)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} - 1 = 0.$$

This h_0 is the height of the boundary of the horizon. Thus we have control outside the horizon. We can also argue that $V(h)$ can only go up, even where V' does not exist.

See figure 3 for this case. We also know $h_{max} - h_0 \leq C_n m^{\frac{1}{n-2}}$. Thus if m is small, we know we're close to flat.

For $n = 3, 4$, we don't expect to get bound on maximal height, since for Schwarzschild, this height is infinite. (This is work in progress.) See figure 4. We get ellipticity of the linearized mean? curvature problem, since $R \geq 0$. Thus, by a comparison principle, the Schwarschild graph has to be above the graph of the other one, if it touches somewhere on the boundary. See fig 5. We are worried about the area of Σ getting large (large ovals), but a small ball still being all that fits. So, for right now, we need to assume some uniformity assumptions, which implies that $V(h_1) \leq dV_0$, where V_0 is the volume of the horizon in Schwarzschild. For volume ODE, $h_1 - h_0 \leq c(d,n)m^{\frac{1}{n-2}}$.

How much control do you think you can get from small mass? Answer: Well, probably need a weak norm, but perhaps outside a compact set you could do better.

Under h_0, that region, do you have a volume bound? No, the estimate on $V'(h)$ doesn't tell us anything.