Fig 1:

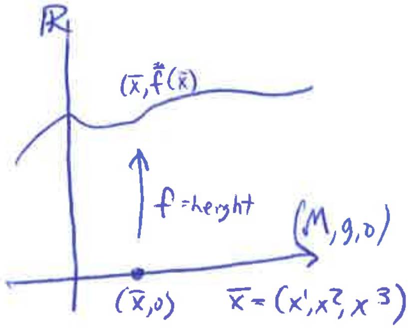


Fig 2

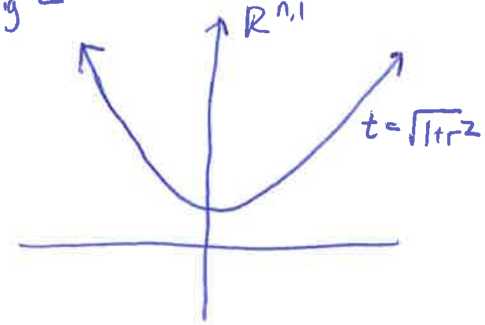


Fig 3

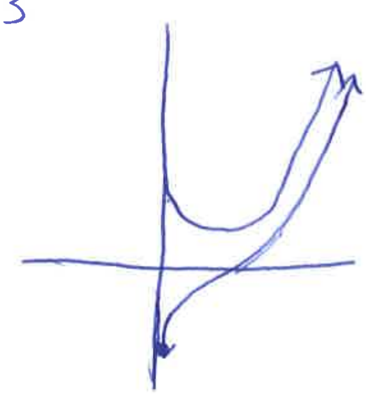
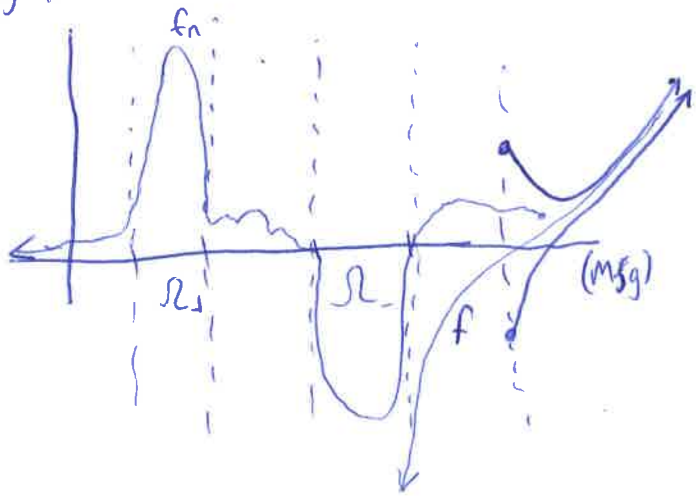


Fig 4



A JANG EQUATION APPROACH TO THE POSITIVE MASS THEOREM FOR ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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1. Spacetime PMT for AE initial data sets

Definition 1.1. The triplet (M, g, K) is an asymptotically Euclidean (AE) initial data set (i.d.) if (M, g) is a complete AE manifold ($M \setminus C \simeq \mathbb{R}^n \setminus B_R$ and $g_{ij} = \delta_{ij} + O(r^{-(n-2)})$) and $K_{ij} = O(r^{-(n-1)})$.

The dominant energy condition (DEC) is $m \geq |J|_g$, where $\mu := \frac{1}{2}(\mathrm{Scal}^g - |K|_g^2 + (\mathrm{tr}_g K)^2)$ and $J := \mathrm{div}^g(K - (\mathrm{tr}_g K) g)$.

The ADM mass is

$$m_{adm} = \frac{1}{2(n-1)\omega_{n-1}} \lim_{R\to\infty} \int_{S_R} (\mathrm{div}^\sigma g - d(tr_\sigma g) D_R)\, d\mu^\sigma.$$

Spacetime PMT (positive mass theorem) conjecture for AE: If (M, g, K) is an AE i.d. satisfying the DEC, then $m_{adm} \geq 0$. Also, $m_{adm} = 0$ iff (M, g, K) is i.d. for Minkowski space.

Strategy of proof (Schoen-Yau '81, $n = 3$): There is a reduction to the case when $K = 0$, i.e. to the Riemannian positive mass theorem (Schoen-Yau '79), in which case the DEC reduces to $\mathrm{Scal}^g \geq 0$. Thus we need to construct $(\tilde{M}, \tilde{g})$ such that $\mathrm{Scal}^{\tilde{g}} \geq 0$ and $\tilde{m}_{adm} \leq m_{adm}$. Then we can just apply the Riemannian positive mass theorem, and get the desired theorem immediately.

Construction: Let $(\bar{M}, \bar{g})$ be a graph of $f : M \to \mathbb{R}$ in the product space $(M \times \mathbb{R}, g + dt^2)$ with the induced metric. See figure 1. We extend the 2nd fundamental form by $K(\partial_t, \cdot) = 0$.

Observation: If $H_{\bar{M}} = \mathrm{tr}_{\bar{M}} K$ and the DEC holds, then $\mathrm{Scal}^{\bar{g}} \geq 2|q|_{\bar{g}}^2 - 2\mathrm{div}^{\bar{g}} q$ where q is some 1 -form measuring the difference between g and k. In terms of f, using the base coordinates, this equation becomes

$$\left(g^{ij} - \frac{f^i f^j}{1 + |df|_g^2}\right)\left(\frac{\mathrm{Hess}_{ij} f}{\sqrt{1 + |df|_g^2}} - K_{ij}\right) = 0$$

This is Jang's equation. We rewrite this as

$$H_g(f) - (\mathrm{tr}_g K)(f) = 0.$$

If $f : M \to \mathbb{R}$ exists and is C^3, then $(\bar{M}, \bar{g})$ is an AE manifold, such that there exists $u > 0$ such that $\tilde{g} = u^4 \bar{g}$ has $\mathrm{Scal}^{\tilde{g}}$ and $\tilde{m}_{adm} \leq m_{adm}$.

An "entire solution" (on the whole manifold) doesn't need to exist, but the so-called geometric solution always does, and so the argument works with suitable modifications.

2. AH i.d. and Positive Mass Conjecture

Intuitively: (M, g) is asymptotically hyperbolic (AH) if $M \setminus C \simeq \mathbb{H}^n \setminus B_R$ and $g \to g_{\mathbb{H}}$ at ∞ sufficiently fast. The model example is the upper unit hyperboloid in Minkowski spacetime. See figure 2. This space sits in $(\mathbb{R}^{n,1}, -dt^2 + dr^2 + r^2 d\sigma)$. We get $g = K = g_{\mathbb{H}} = \frac{dr^2}{1+r^2} + r^2 d\sigma$.

Definition 2.1. (M, g, K) is an AH initial data set if (M, g) is a complete AH manifold, $M \setminus C \simeq S^{n-1} \times (R, \infty)$ such that $g = \frac{dr^2}{1+r^2} + r^2(\sigma + \frac{m}{r^n} + h)$ where m is a 2 tensor on the unit sphere (that does not depend on r) called the mass aspect tensor. Also, h is an r dependent family of 2-tensors on S^{n-1}, with $h_{\alpha\beta} = O(r^{-(n+1)}$. We also require $K = g + \gamma$, $|\gamma|_{g_{\mathbb{H}}} = O(r^{-(n+1)})$. Thus the leading order terms in g and K are the same. Note that umbilic initial data fits into this.

Spacetime positive mass conjecture for AH initial data: If (M, g, K) is an AH i.d. satisfying DEC, then

$$m_{AH} := c_n \int_{S^{n-1}} (\mathrm{tr}_\sigma m) d\mu^\sigma \geq 0$$

and $m_{AH} = 0$ if and only if (M, g, K) is i.d. for a Minkowski spacetime.

Results available: Riemannian PMT: $(K = g)$. If (M, g) is a complete AH manifold such that $\mathrm{Scal}^g \geq -n(n-1)$ (DEC) then $m_{AH} \geq 0$ and M_{AH} if and only if it is $\mathbb{H}^n$. This has been proven by Wang (2001) and Chrusciel-Herzlich (2003) under spin assumption, but in all dimensions. Andersson-Cai-Galloway (2008) proved it for $3 \leq n \leq 7$ and $\mathrm{tr}_\sigma m$ has a fixed sign (¿0, ¡0 or =0). The general conjecture has been addressed by Zhang for $n = 3$ and spin assumption.

3. Jang's equation reduction of the spacetime PMT for AH i.d. sets

This is my own work in progress. It is inspired by ideas from "Bondi mass is positive" from Schoen-Yau '81.

In the model case, $g = k = g_{\mathbb{H}}$, the solution of Jang's equation is $f = \sqrt{1 + r^2}$. Clearly the induced metric is $\bar{g} = g + df \otimes df = dr^2 + r^2\sigma = \delta$. So this reduces to the Euclidean case! This is in agreement with our previous observation that $\mathrm{Scal}^{\bar{g}} \geq 2|q|^2 - 2\mathrm{div} q$. So, let $f = \sqrt{1 + r^2} + \alpha(\theta, \phi) \ln r + \psi(\theta, \phi) + o(1)$ and plug it into Jang's equation. We get

$$\alpha = const = \frac{3}{8\pi} \int_{S^2} (\mathrm{tr}_\sigma) m d\mu^\sigma = 2m_{AH},$$

$$\Delta^{S^2} \psi = -\alpha + \frac{3}{2} \mathrm{tr}_\sigma m.$$

This tells us that we should look for a solution in the above form, with the above forms for α and ψ.

To get this type of asymptotics, we need to construct barriers with those asymptotics. Barriers: For an upper barrier we pick $f_+ : \{r \geq R\} \to \mathbb{R}$ such that it is a supersolution of Jang's equation, and $\partial f_+/\partial r|_{r=R} = -\infty$. See fig 3. We similarly pick lower barrier. (In the AE case, $f \to 0$ implies that the barriers are easy to find.)

We'll look for barriers in the from $f_+ = \varphi(r) + \psi(\theta, \phi)$. Jang's equation becomes a Riccati type ODE modulo some correction terms.

Geometric solution: We solve the following system:

$$H_g(f_n) - \mathrm{tr}_g(K)(f_n) = \tau_n f_n$$

on $\{r \leq R_n\}$ and $f_n = \varphi_n$ on the boundary, $\{r = R_n\}$ where $\tau_n \to 0, R_n \to \infty$ and $f_- \leq \varphi_n \leq f_+$. See figure 4. We get these f_n, but they may blow up on some sets $\Omega_\pm$, and so we may not have a proper solution, but we have a geometric solution. There is a Harnack inequality for these solutions, which says that each component is either a graph or a cylinder.

Assume $f : M \to \mathbb{R}$ exists with no blowups. Let $(\bar{M}, \bar{g})$ be the graph of f. We need to show that it is an AE manifold. In the AE case, this can be shown just by looking at $g + df \otimes df$, using rescaling. Rescaling doesn't work on AH manifolds. However, we can write our manifold as a graph over the manifold given by f_-, and can use this to get the argument to work.

We can also show that $\bar{m}_{adm} = 2m_{AH}$. Let $\tilde{g} = u^4\bar{g}$, $u = 1 + \beta/r + O(r^{-2})$ where $\beta \leq -\alpha/4$ and $\mathrm{Scal}^{\tilde{g}} = 0$. We then have

$$0 \leq \tilde{m}_{adm} = \bar{m}adm + 2\beta \leq 2m_{AH} + \alpha/2 = m_{AH}$$

which completes the proof.