Lu Wang - Topology of Closed Hypersurfaces of small entropy

Joint work with Jacob Bernstein

Lets consider Euclidean space $(\mathbb{R}^{n+1}, \delta_{ij}, e^{-|x|^2/4})$ with Gaussian probability

Now, consider a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$

Define the Gaussian surface of $\Sigma$

$$F[\Sigma] = \underbrace{(4\pi)^{-n/2}} \int_\Sigma e^{-|x|^2/4}$$

want to multiply by this constant b/c if we evaluate the hypersurface we want

$F[\mathbb{R}^n] = 1$


Euler Lagrange eqn

$$(*) \quad H_\Sigma = \frac{1}{2} \langle x, \vec{n}_\Sigma \rangle \qquad H_\Sigma = \operatorname{div}_\Sigma \vec{n}_\Sigma$$

If $\Sigma$ ~~&~~ satisfies $(*)$, then $\Sigma$ is called a <u>self shrinker</u>

A Natural question to ask is:

(1) What is the smallest gaussian surface area among all these self shrinkers?

One can answer this question with geometric flows. In particular, with mean curvature flow (MCF)
this is of course not the only way to answer this question.

What do I mean by MCF?

A 1-parameter family of hypersurfaces $M_t^n \subset \mathbb{R}^{n+1}$ moves by mean curvature if

$$(\partial_t x)^\perp = \vec{H}_{M_t} \text{ for } x \in M_t$$

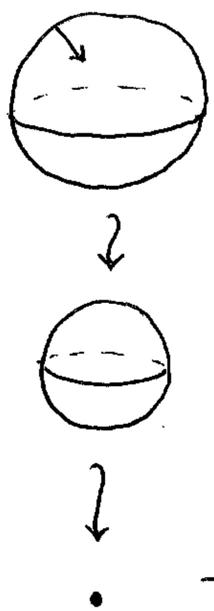
$$\vec{H}_{M_t} = - H_{M_t} \cdot \vec{n}_{M_t}$$

EXAMPLES:

(1) $M_0$ is minimal (i.e. $\vec{H}_{M_0} \equiv 0$)

$M_t = M_0 \quad \forall t$ hypersurface is not going to move under the flow

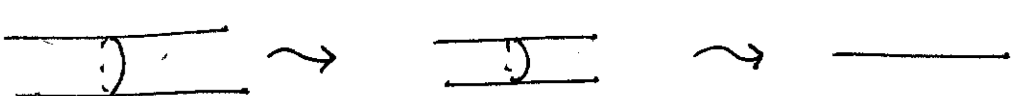
(2) $S^n \subset \mathbb{R}^{n+1}$



$T < \infty$

Sphere will remain spherical under the flow and will shrink to a point as the curvature of $S^n$ goes to $\infty$.

(3) $S^k \times \mathbb{R}^{n-k}$ cylinder, cross section shrinks to a point



$T < \infty$

In general, MCF starting from a closed (compact, without boundary) hypersurface will have curvature blow up in finite time.

(4) If $\Sigma$ is a ~~hypersurface~~ self-shrinker in $\mathbb{R}^{n+1}$

i.e $H_{\Sigma} = \frac{1}{2} \langle X, \vec{n}_{\Sigma} \rangle$

Define family $\Sigma_t = \sqrt{-t}\, \Sigma$ to be a scaling of $\Sigma$.

$\{\Sigma_t\}_{t<0}$ moves by mean curvature, i.e. for $x \in \Sigma_t$,

$$(\partial_t X)^{\perp} = \vec{H}_{\Sigma_t}$$

Now we want to answer question (1); to do that we need to talk about the following results:

★ Huiskens Monotonicity formula

If $\{M_t\}$ is a MCF, then

$$\frac{d}{dt} \int_{M_t} \Phi_{(x_0,t_0)} = - \int_{M_0} \left| \vec{H}_{M_t} + \frac{(x-x_0)^{\perp}}{2(t_0-t)} \right|^2 \Phi_{(x_0,t_0)}$$

for an arbitrary point $x_0 \in \mathbb{R}^{n+1}$, $t \in \mathbb{R}$, $t<t_0$.

$$\Phi_{(x_0,t_0)}(x,t) = (4\pi(t_0-t))^{-n/2} e^{|x-x_0|^2/(4(t-t_0))}$$

★ Colding & Micozi introduced the <u>entropy</u> of a hypersurface $\Sigma$,

$$\lambda[\Sigma] = \sup_{\substack{y \in \mathbb{R}^n \\ \rho > 1}} F[\rho\Sigma + y]$$

sup is over translations + ~~rotations~~ scalings of $\Sigma$.

PROPOSITION

(1) If $\{M_t\}$ is a MCF
$t \longmapsto \lambda[M_t]$ is a decreasing function

(2) If $\Sigma$ is a self shrinker, then
$$F[\Sigma] = \lambda[\Sigma] \quad (\rho = 1,\ y = 0)$$

If it (i. $\Sigma$) is not compact it is not clear sup can be achieved, if compact it is not clear when sup is achieved, but if a self-shrinker then we do know.

Now we can address question (1):

Brakke's local regularity theorem (White's version)
(holds for large class of MCF, but we aren't looking at the general thm).

$\exists$ a constant $\delta(n) > 0$ s.t. if
$$\lambda[M_0] < 1 + \delta(n)$$
then $\{M_t\}$ a MCF starting from $M_0$ will have uniform curvature bound away from time zero.

Take any self shrinker, $\Sigma^n \subset \mathbb{R}^{n+1}$, and assume $\Sigma^n \neq \mathbb{R}^n$. If $\lambda[\Sigma^n] = F[\Sigma^n] < 1 + \delta(n)$
$$\Rightarrow \{ \Sigma_t = \sqrt{-t}\,\Sigma \} \text{ a MCF.}$$

By Brakke's local regularity thm $\{\Sigma_t\}$ has bounded curvature up to $t = 0$.

CONJECTURE 2 (CIMCW)

For $2 \le n \le 6$, the rand sphere has the smallest entropy among all closed hypersurfaces of $\mathbb{R}^{n+1}$

( $$\lambda[\Sigma] = \sup_{\substack{y \in \mathbb{R}^{n+1} \\ \rho > 0}} F[\rho\Sigma + y]$$

is invariant under scalings + translations

You can answer conj 2 without conj 1 even though conj 1 $\Rightarrow$ conj 2.

THEOREM (Bernstein - Wu '14)

Conjecture #2 is true.

Remark: did not prove conj 1.

REMARK: Min max approach to study conj 2. Minimal hypersurface in $S^n$ you can use stereographical projection

$$\left( \mathbb{R}^n, \frac{1}{(1+|x|^2)^n} \right)$$

THEOREM: (Ketwer - Zhen '15)

Conjecture #2 is true for $n=2$ and genus $\neq 1$

Proof is highly analogous to Marques-Neves proof of Willmore conjecture.

Question 4: What about closed hypersurfaces with entropy close to that of the rand sphere?

THEOREM (Bernstein - W '15)

For $n = 2, 3$ If $\Sigma^n$ is a closed hypersurface of $\mathbb{R}^{n+1}$ and $\lambda[\Sigma^n] \le \lambda[S^{n-1}]$ then $\Sigma^n$ is diffeomorphic to $S^n$

This leads to a contradiction as $\{\Sigma_t\}$ has curvature blow up at $t=0, x=0$ (b/c we assumed $\Sigma^n \neq \mathbb{R}^n$). Thus either $\Sigma^n = \mathbb{R}^n$ or

$$\lambda[\Sigma^n] \geq 1 + \delta(n).$$

$\Rightarrow$ the plane $\mathbb{R}^n$ has the smallest Gaussian surface area among all self shrinkers. Moreover, $\exists$ a gap between $\mathbb{R}^n$ and the ~~hypers~~ self shrinker with second smallest gaussian surface area.

Question (2): What is the 2nd smallest Gaussian surface area among all self-shrinkers?
What's the optimal constant $\delta(n)$?

CONJECTURE 1 (Colding-Ilmanen-Minicozzi-White)
For $2 \leq n \leq 6$, $S^n(\sqrt{2n})$ has the smallest entropy among all non-flat self-shrinkers.

$\sqrt{2n}$: radius of sphere

$n=1$, not interesting
$n > 6$, sphere won't necessarily work anymore (simmons cone)

Colding-Minicozzi classified all entropy stable self shrinkers. i.e $S^n(\sqrt{2n})$, $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, $\mathbb{R}^n$.

THEOREM (C-I-M-W 13)
Conjecture 1 is true for all closed self shrinkers

REMARK $\lambda[S^{n-1}] > \lambda[S^n]$

Key is to study topology of non-compact self-shrinkers with flow entropy.

THEOREM 3 (Bernstein-W '15)
For $n \geq 2$, if $\Sigma^n$ is an asymptotically conical self shrinker and

$$\lambda[\Sigma^n] \leq \Sigma[S^{n-1}]$$

$\Rightarrow \Sigma$ is contractible and the — of the asympto cone is a homology sphere.

COROLLARY $n=2$, $\Sigma$ as in thm 3 is topologically $\mathbb{R}^2$ with Brendle's uniqueness thm or genus 0 self shrinkers, $\Sigma$ is flat, i.e $\Sigma = \mathbb{R}^2$. $n=3$, $\Sigma$ is diffeomorphic to $\mathbb{R}^3$ (by Alexanders thm)

More ambitous questions:

* Can one prove a ~~self shrinker~~ schverfles type theorem for closed hypersurfaces with low entropy? i.e. the region enclosed by $\Sigma$ is diffeo to $D^{n+1}$

$n=3$.

~~Questions~~