Ancient Solutions to Geometric Flows

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Ancient and Eternal Solutions

- We will discuss ancient or eternal solutions to geometric parabolic partial differential equations.
- These are special solutions which exist for all time

$$-\infty < t < T \qquad \text{where} \;\; T \in (-\infty, +\infty].$$

- They appear as blow up limits near a singularity.
- Understanding ancient and eternal solutions often sheds new insight to the singularity analysis

Outline

In this talk we will address:

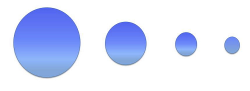
- the classification of ancient solutions to parabolic partial differential equations, with emphasis to geometric flows: Mean Curvature flow, Ricci flow and Yamabe flow.
- methods of constructing new ancient solutions from the gluing of two or more solitons (self-similar solutions).
- new techniques and future research directions.

Ancient and Eternal solutions

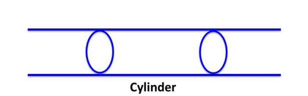
- Definition: A solution $u(\cdot, t)$ to a parabolic equation is called ancient if it is defined for all time $-\infty < t < T$, $T < +\infty$.
- Ancient solutions typically arise as blow up limits at a type I singularity.
- Definition: A solution $u(\cdot, t)$ to a parabolic equation is called eternal if it is defined for all $-\infty < t < +\infty$.
- Eternal solutions as blow up limits at a type II singularity.

Solitons

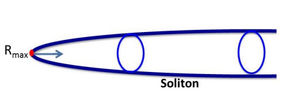
- Solitons (self-similar solutions) are typical examples of ancient or eternal solutions and often models of singularities.
- Some typical examples of solitons to geometric PDE are:
- Spheres:

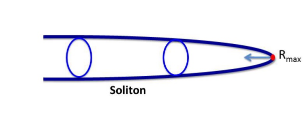


- Cylinders:



- Translating solitons:



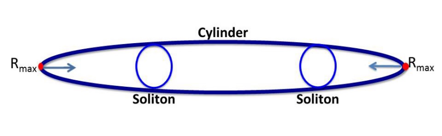


Solitons as singularities

- A well known technique introduced by R. Hamilton (1995) has been widely used to characterize as solitons the eternal solutions to geometric flows which attain a space-time curvature maximum.
- Such solutions typically appear as *carefully chosen* blow up limits near type II singularities.
- Its proof relies on a clever combination of the strong maximum principle and Li-Yau type differentiable Harnack estimates.

Other Ancient and eternal solutions

- However, there exist other ancient or eternal solutions which are not solitons.
- These, often may be visualized as obtained from the gluing as $t \to -\infty$ of two or more solitons.



- Obtaining more information about such solutions, often leads to the better understanding of the singularities.
- Objective: How to construct such solutions and how to characterize them among all ancient or eternal solutions.

Geometric conditions of ancient or eternal solutions

Goal: Characterize all ancient or eternal solutions to a geometric flow under natural geometric conditions:

- Being a soliton (self-similar solution)
- Satisfying an appropriate curvature bound as $t \to -\infty$:
  - i. Type I: global curvature bound after typical scaling.
  - ii. Type II: solutions which are not type I.
- Satisfying a non-collapsing condition.

Liouville's theorem for the heat equation on manifolds

- Let M^n be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with Ricci $(M^n) \geq 0$.
- Yau (1975): Any positive harmonic function u on M^n must be constant.
- This is the analogue of Liouville's Theorem for harmonic functions on $\mathbb{R}^n$.
- Question: Does the analogue of Yau's theorem hold for positive solutions of the heat equation

$$u_t = \Delta u \qquad \text{on } M^n?$$

- Answer: No. Example $u(x,t) = e^{x_1+t}$, $x = (x_1, \cdots, x_n)$ on $M^n := \mathbb{R}^n$.

A Liouville type theorem for the heat equation

- Souplet - Zhang (2006): Let M^n be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with Ricci $(M^n) \geq 0$.

  (a) If u be a positive ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that

  $$u(p,t) = e^{o(d(p)+\sqrt{|t|})} \qquad \text{as } d(p) \to \infty$$

  then u is a constant.

  (b) If u be an ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that

  $$u(p,t) = o(d(p) + \sqrt{|t|}) \qquad \text{as } d(p) \to \infty$$

  then u is a constant.

- Proof: By using a local gradient estimate of Li-Yau type on large appropriately scaled parabolic cylinders.

Semilinear equations on $\mathbb{R}^n$

- Consider positive solutions $u > 0$ of the similinear elliptic equation
$$\Delta u + f(u) = 0, \qquad \text{on } \mathbb{R}^n.$$
- Well known example related to the Yamabe problem is $f(u) = u^{\frac{n+2}{n-2}}$.
- Gidas, Ni and Nirenberg (1979): Solutions $u > 0$ with mild decay condition as $|x| \to +\infty$ are rotationally symmetric.
- Many related important subsequent results including those by Cafarelli, Gigas and Spruck and Berestycki and Nirenberg.

The Semi-linear heat equation

- Consider next the semilinear heat equation

$$(\star_{SL}) \quad u_t = \Delta u + u^p \quad \text{on } \mathbb{R}^n \times (0, T)$$

  in the subcritical range of exponents $1 < p < \frac{n+2}{n-2}$.

- It provides a prototype for the blow up analysis of geometric flows.
- In particular in neckpinches of solutions to the Ricci flow and Mean Curvature flow.
- Also in the characterization of rescaled limits as $t \to -\infty$ of ancient solutions.

The rescaled semi-linear heat equation

- Self-similar scaling at a singularity at (a, T):

$$w(y,\tau) = (T-t)^{\frac{1}{p-1}}\, u(x,t), \quad y = \frac{x-a}{\sqrt{T-t}}, \quad \tau = -\log(T-t).$$

- Giga - Kohn (1985): $\|w(\tau)\|_{L^\infty(\mathbb{R}^n)} \le C, \ \tau > -\log T.$
- The rescaled solution satisfies the equation

$$(\star) \qquad w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p.$$

- Objective: To analyze the blow up behavior of u one needs to understand the long time behavior of w as $\tau \to +\infty$.
- This is closely related to the classification of bounded eternal solutions of $(\star)$.

Eternal solutions of the semi-linear heat equation

- Problem: Provide the classification of bounded positive eternal solutions w of equation

$$(\star) \qquad w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p.$$

- Eternal means that $w(\cdot, \tau)$ is defined for $\tau \in (-\infty, +\infty)$.
- The only steady states of $(\star)$ are the constants:

$$w = 0 \quad \text{or} \quad w = \kappa, \quad \text{with} \;\; \kappa := (p-1)^{-\frac{1}{(p-1)}}.$$

- Theorem (Giga - Kohn '87) $\lim_{\tau \to \pm\infty} w(\cdot, \tau) =$ steady state.
- Space independent eternal solutions : $\phi(\tau) = \kappa(1 + e^\tau)^{-\frac{1}{(p-1)}}$.

Classification of Eternal solutions

- Theorem (Giga - Kohn '87 and Merle - Zaag '98)
If w is bounded positive eternal solution of $(\star)$ defined on $\mathbb{R}^n \times (-\infty, +\infty)$, then

$$w = 0 \quad \text{or} \quad w = \kappa \quad \text{or} \quad w = \phi(\tau - \tau_0).$$

- Main difficulty (Merle - Zaag): Classify the orbits $w(\cdot, \tau)$ that connect the two steady states:

$$\lim_{\tau \to -\infty} w(\cdot, \tau) = \kappa \quad \text{and} \quad \lim_{\tau \to +\infty} w(\cdot, \tau) = 0$$

- Important Liouville type results related to equation $(\star_{SL})$ by: P. Polacik, P. Quittner, T. Bartsch, P. Souplet, E. Yanagida among others.

Liouville type results for solutions to parabolic equations

- Although Liouville type results often appear with respect to elliptic equations, there are not many such results available in the parabolic setting.
- G. Koch, N. Nadirashvili, G. Seregin and V. Sverak (2009):
  (i) Liouville type result for ancient bounded solutions $u(x,t)$ of the 2-dim Navier Stokes equations.

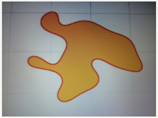
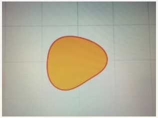
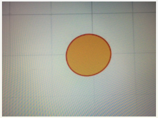
  (ii) Also, similar result for bounded, axi-symmetric with no swirl solutions $u(x,t)$ of the 3-dim Navier Stokes equations.

Ancient Convex solutions to the CSF

- Let Γ_t be a family of closed curves which is an embedded solution to the Curve shortening flow, i.e. the embedding $F : \Gamma_t \to \mathbb{R}^2$ satisfies

$$\frac{\partial F}{\partial t} = -\kappa \, \nu$$

with κ the curvature of the curve and ν the outer normal.

- Gage (1984); Gage and Hamilton (1996); Grayson (1987): the CSF shrinks Γ_t to a round point.
- Problem: Classify the ancient compact embedded solutions to the Curve shortening flow.

Ancient Convex solutions to the CSF

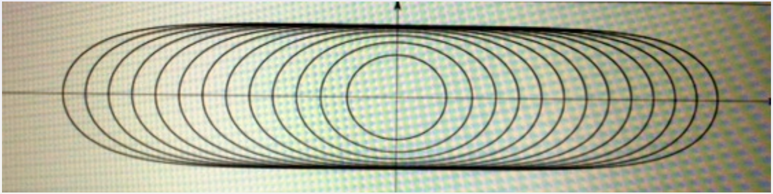
- The curvature κ of Γ_t evolves, in terms of its arc-length s, by

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

- Definition: Γ_t is type I if $\limsup_{t \to -\infty} \sqrt{|t|} \max_{\Gamma_t} \kappa(\cdot, t) < \infty$. Otherwise, Γ_t is of type II.
- Type I solution: the contracting circles.
- Type II solution: the Angenent ovals (paper clips). These are ancient convex solutions in closed form which are not solitons.

The Classification of Ancient Convex solutions to the CSF

- The Angenent ovals (paper clips) as $t \to -\infty$ may be visualized as two grim reaper solutions glued together.



- Theorem (D., Hamilton, Sesum - 2010)
  The only ancient convex solutions to the CSF are the contracting spheres or the Angenent ovals.

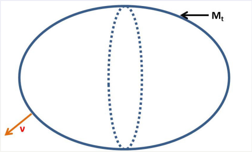
- Proof: It is based on various monotonicity formulas and the the fact that at its singular time any solution becomes circular with *very sharp rates of convergence*.

Non-Convex ancient solutions

- Question: Do they exist non convex compact embedded solutions to the curve shortening flow ?
- Angenent (2011): Presents a YouTube video of an ancient solution to the CSF built out from one Yin-Yang spiral and one Grim Reaper.

- S. Angenent: is currently working on a rigorous construction of these solutions.

- Let M_t, $t \in (-\infty, T)$ be a smooth ancient compact solution of the Mean curvature flow



$$\text{(MCF)} \qquad \frac{\partial F}{\partial t} = -H\,\nu$$

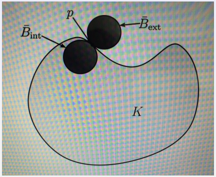
$H(p,t)$ is the Mean curvature and ν a choice of unit normal.

- Problem: Understand ancient compact solutions M_t of the Mean curvature flow.

Ancient non-collapsed solutions to MCF

- Weimin Sheng and Xu-Jia Wang; Ben Andrews: Introduced an α-noncollapsed condition.

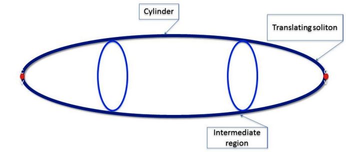
$B = B_{\frac{\alpha}{H(p)}}$



- Haslhofer & Kleiner (2013):
  Ancient compact + α-noncollapsed MCF solution $\Rightarrow$ convex.
- convex compact + self-similar MCF solution $\Rightarrow$ sphere.
- Ancient ovals: Any compact and α-noncollapsed solution to MCF which is not self-similar.
- Other ancient solutions to MCF: compact and collapsed.

Ancient MCF ovals

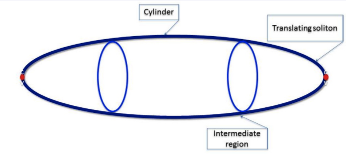
- Problem: Provide the classification of all Ancient ovals.
- B. White (2003): Existence of certain Ancient ovals with $O(k) \times O(l)$ symmetry. We call them White ancient ovals.
- Haslhofer & Hershkovits (2013): Give more details in the existence proof of the White Ancient ovals.
- Angenent (2012): establihes the formal matched asymptotics of all Ancient ovals as $t \to -\infty$.



- They are small perturbations of ellipsoids.

Unique asymptotics of Ancient MCF ovals

- S. Angenent, D., and N. Sesum (2015): All ancient ovals with $O(1) \times O(n)$ symmetry have unique asymptotics as $t \to -\infty$, and satisfy Angenent's precise matched asymptotics:



- Geometric properties $t \to -\infty$: type II ancient solutions

$$\mathrm{diam}(t) \approx \sqrt{8|t| \log |t|} \quad \text{and} \quad H_{\max}(t) \approx \frac{\sqrt{\log |t|}}{\sqrt{2|t|}}.$$

- The proof involves: the analysis of the linearized operator at the cylinder, Huisken's monotonicity formula and carefully constructed barriers at the intermediate region.

Uniqueness of Ancient MCF ovals

- Work in progress: to establish such asymptotics in the non-symmetric case.
- Next Step: Establish the uniqueness of the Ancient ovals.
- Conjecture 1: The Ancient ovals with $O(l) \times O(k)$ symmetry are uniquely determined by their asymptotics at $t \to -\infty$.
- Hence: they are unique (up to dilation and translation in rescaled time).
- Conjecture 2: All Ancient ovals are $O(l) \times O(k)$ symmetric.

Ancient compact solutions to the 2-dim Ricci flow

- Consider an ancient solution of the Ricci flow

$$\text{(RF)} \qquad \frac{\partial g_{ij}}{\partial t} = -2\, R_{ij}$$

  on a compact manifold M^2 which exists for all time $-\infty < t < T$ and becomes singular at time T.

- In dim 2, we have $R_{ij} = \frac{1}{2} R\, g_{ij}$, where R is the scalar curvature.

- Hamilton (1988), Chow (1991): After re-normalization, the metric becomes spherical at $t = T$.

- Problem: Provide the classification of all ancient compact solutions.

Ancient compact solutions to the 2-dim Ricci flow

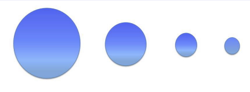
- Choose a parametrization $g_{S^2} = d\psi^2 + \cos^2\psi \, d\theta^2$ of the limiting spherical metric.
- We parametrize our solution as $g(\cdot, t) = u(\cdot, t) \, g_{S^2}$.
- Then the (RF) becomes equivalent to the fast-diffusion equation:

$$u_t = \Delta_{S^2} \log u - 2, \quad \text{on } S^2 \times (-\infty, T).$$

- Provide the classification of all ancient solutions.

Examples of compact solutions on S^2

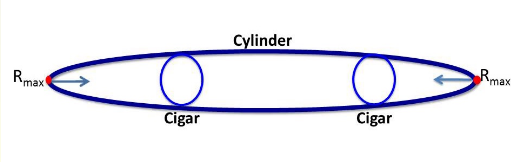
- Type I solution: the contracting spheres.



- Type II solution: the King-Rosenau solution of the form:

$$u(\psi, t) = [a(t) + b(t)\sin^2\psi]^{-1}, \quad t < T.$$

As $t \to -\infty$ the King-Rosenau solution looks like two cigar solitons glued together.



The classification result

Theorem: (D., Hamilton, Sesum - 2012)

The only ancient solutions to the Ricci flow on S^2 are the contracting spheres and the King-Rosenau solutions.

Proof: combines geometric arguments and PDE techniques.

i. a monotonicity formula and uniform a priori $C^{1,\alpha}$ estimates that allow us to pass to the limit as $t \to -\infty$.

ii. geometric arguments that allow us to classify the backward limit as $t \to -\infty$.

iii. maximum principle arguments that allow us to characterize the King-Rosenau solutions among type II solutions.

iv. an isoperimetric inequality that allows us to characterize the contracting spheres among type I solutions.

The characterization of King solutions

- To capture the King solutions we consider the scaling invariant nonotone quantity

$$Q(x,y,t) := \bar{v}\left[\left(\bar{v}_{xxx} - 3\bar{v}_{xyy}\right)^2 + \left(\bar{v}_{yyy} - 3\bar{v}_{xxy}\right)^2\right]$$

where $\bar{v} := \bar{u}^{-1}$ is the pressure in plane coordinates.

- Using complex variable notation $z = x + iy$, this quantity is nothing but

$$Q = \bar{v}\,|\bar{v}_{zzz}|^2.$$

- The quantity Q is well defined.
- It turns out that $Q \equiv 0$ implies that $\bar{v}$ is one of the King solutions.
- To establish that $Q \equiv 0$ we prove that:

  i. $Q_{\max}(t)$ is decreasing in t (by considering its evolution equation), and

  ii. $\lim_{t\to-\infty} Q_{\max}(t) = 0$.

The 3 dimensional Ricci flow - Open problems

- 3-dim Ricci flow: The analogue of the 2-dim King-Rosenau solutions have been shown to exist by G. Perelman. They are not given in closed form, they are type II and k-noncollapsed.
- Other collapsed compact solutions in closed form have been found by V.A. Fateev in a paper dated back to 1996.
- Conjecture: The only k-noncollapsed ancient and compact solutions to the 3-dim Ricci flow are the contracting spheres and the Perelman solutions.
- Brendle, Huisken & Sinestrari (2011): Present a pinching curvature condition that *characterizes* the ancient compact solutions to the 3-dim Ricci as contracting spheres.

Ancient solutions to the Yamabe flow

- We will conclude by discussing ancient solutions $g = g_{ij}$ of the Yamabe flow on S^n, $n \geq 3$.
- The Yamabe flow may be viewed as the higher dimensional analogue of the 2-dim Ricci flow.
- It is the evolution of metric $g(\cdot, t)$ conformally equivalent to the standard metric on S^n by

$$\frac{\partial g}{\partial t} = -R\, g \qquad \text{on} \quad -\infty < t < T$$

where R denotes the scalar curvature of g.

- Question: Is it possible to provide the classification of all such ancient solutions ?

The Yamabe flow - Background

- Let (M^n, g_0), $n \geq 3$ be a compact manifold without boundary. The scalar curvature R of a metric $g = v^{\frac{4}{n-2}} g_0$ conformal to g_0 is given by

$$R = -v^{-\frac{n+2}{n-2}} \left(c_n \Delta_{g_0} v - R_0 \, v \right)$$

  where R_0 denotes the scalar curvature of g_0.

- R. Hamilton (1989): introduced the Yamabe flow as a parabolic approach to resolve the Yamabe problem.

- S. Brendle (2007): convergence of the normalized flow to a metric of constant scalar curvature (up to a mild technical assumption for dim $n \geq 6$).

- Previous important works: Hamilton '89, Chow '92, Ye '94, Schwetlick-Struwe '2003.

Ancient solutions to the Yamabe flow on S^n

- Let $g = v^{\frac{4}{n-2}}\, g_{S^n}$ be an ancient solution to the Yamabe flow, which is conformal to the standard metric on S^n.
- The function v evolves by the fast diffusion equation

$$(v^{\frac{n+2}{n-2}})_t = \Delta_{S^n} v - c_n\, v \quad \text{on } S^n \times (-\infty, T).$$

- Let $g = \bar{v}^{\frac{4}{n-2}}\, g_{\mathbb{R}^n}$ after stereographic projection. Then,

$$(\bar{v}^{\frac{n+2}{n-2}})_t = \Delta \bar{v} \quad \text{on } \mathbb{R}^n \times (-\infty, T).$$

- Definition: An ancient solution is called of type I if:

$$\limsup_{t \to -\infty} \big(|t| \max_{S^n} |Rm|(\cdot, t) \big) < \infty.$$

Otherwise, it is called of type II.

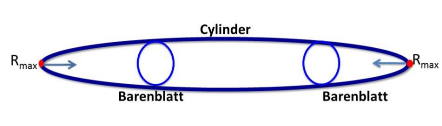
The King Solutions

- J.King (1993): discovered non-self similar type I ancient compact solutions to the (YF) on S^n in closed form.
- King solutions: $g = \hat{v}_K(\cdot, t)^{\frac{4}{n-2}} \, g_{\mathbb{R}^n}$, where

$$\hat{v}_K(x,t) = \left(a(t) + 2b(t)\,|x|^2 + a(t)|x|^4\right)^{-\frac{n-2}{4}}, \qquad x \in \mathbb{R}^n.$$

- As $t \to -\infty$ they converge (after rescaling) to two Barenblatt type self-similar solutions (shrinking solitons) joined by a long cylindrical neck.

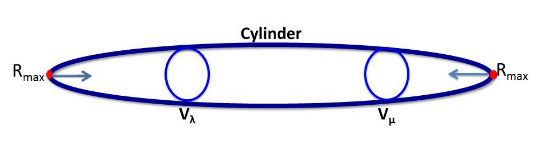
$t \approx -\infty$



- Question 1:
  Are the contracting spheres and the King solutions the only examples of type I ancient solutions ?
- Question 2:
  Are there any type II ancient solutions ?

New Type I solutions to the Yamabe flow

- Recent work: (D., del Pino, J. King and N. Sesum - 2015) There exist infinite many other type I ancient solutions.
- As $t \to -\infty$ they look as two self-similar solutions v_λ, v_μ connected by a cylinder and moving with speeds $\lambda > 0, \mu > 0$.



- Our solutions are not given in closed form but we show very sharp asymptotics.
- In similar spirit to the work by Hamel and Nadirashvili (1999) where they construct ancient solutions for the KPP equation

$$u_t = u_{xx} + f(u), \qquad x \in R.$$

Shrinking solitons with cylindrical behavior

- We look for rotationally symmetric shrinking solitons of the (YF) expressed in cylindrical coordinates $g = v^{\frac{4}{n-2}} g_{cyl}$.
- $v(x, \tau)$ satisfies (after a type I rescaling) the equation:

$$(*) \qquad (v^{\frac{n+2}{n-2}})_\tau = v_{xx} - v + v^{\frac{n+2}{n-2}}.$$

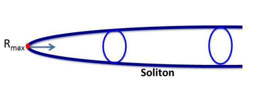
- Shrinking solitons (or traveling waves): $\forall \lambda \geq 1$ there exist a solution $v_\lambda = V_\lambda(x - \lambda\tau)$ of $(*)$ with cylindrical behavior

$$V_\lambda(x) \approx 1 - C_\lambda e^{-\gamma_\lambda x}, \quad \text{as } x \to +\infty.$$

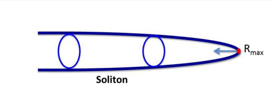
- Theorem: (D., J. King and N. Sesum)
L^1 stability of the traveling wave solutions v_λ.

Shrinking solitons with cylindrical behavior

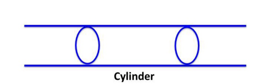
- Consider shrinking solitons in cylindrical coordinates and after a type I scaling.
- Traveling wave to the right: $v_{\lambda,h} = V_\lambda(x - \lambda\tau + h)$



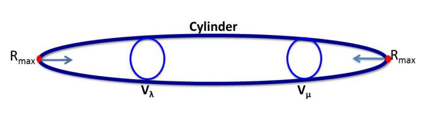
- Traveling wave to the left: $\bar{v}_{\mu,h'} = V_\mu(-x - \mu\tau + h')$



- Clylinder: $\xi_k(\tau) \approx 1 - k\, e^{\tau/2}$, as $\tau \to -\infty$.



- **Theorem:** (D., del Pino, J. King and N. Sesum)
  There exist a five parameter family $v_{\lambda,\mu,h,h',k}$ of type I ancient solutions of the Yamabe flow on $S^n \times (-\infty, T)$.



- In terms of the pressure function $f := v^q$, $q := -\frac{4}{n-2}$ it satisfies:

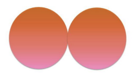
$$v^q_{\lambda,\mu,h,h',k} \approx v^q_{\lambda,h}(x,\tau) + \xi_k(\tau)^q + \bar{v}^q_{\mu,h'}(x,\tau).$$

- **Proof:** By the construction of precise ancient barriers.

Ancient towers of moving bubbles - type II solutions

- Question: Are there any type II ancient solutions to (YF) ?
- D., del Pino and Sesum (2013):
  We construct a class of ancient solutions of the Yamabe flow on S^n which (after re-normalization) converge as $t \to -\infty$ to a tower of n-spheres. They are rotationally symmetric.

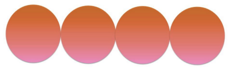
$t \to -\infty$



$t > -\infty$

- The curvature operator in these solutions changes sign and they are of type II.
- Our construction also holds for any number of bubbles.

$t \to -\infty$



Discussion on parabolic gluing methods

- Our construction may be viewed as a parabolic analogue of the elliptic gluing technique.
- Elliptic gluing: pioneering works by Kapouleas '90 -'95 and by Mazzeo, Pacard, Pollack, Ulhenbeck among many others.
- Brendle & Kapouleas (2014): construct new ancient compact solutions to the 4-dim Ricci flow by parabolic gluing.
- Future research direction: apply parabolic gluing on other geometric flows.

Conclusion

- We discussed ancient solutions to geometric parabolic PDE.
- Typical examples are either solitons or other special solutions obtained from the gluing as $t \to -\infty$ of solitons.
- The only existing classification results heavily rely on knowing the exact form of these ancient solutions.
- Future research direction: develop new techniques that allow us to characterize and construct other types of ancient or eternal solutions.