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Monge-Ampère equations on quasi-projective varieties. (Joint work with Chinh Lu (Centro di Giorgi))

Let (X^N, ω) be a compact kahler manifold

$\omega \downarrow$ Kahler form (strict pos, closed real (1,1)-form

$$\omega \overset{loc}{=} \frac{i}{\pi}\sum_{\alpha,\beta} \omega_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta$$

express Ricci form of ω locally

$$Ric(\omega) \overset{loc}{=} \frac{-i}{\pi}\sum_{\alpha,\beta} \frac{\partial^2}{\partial z_\alpha \bar{z}_\beta}(\log\det(\omega_{p\bar{q}}))dz_\alpha \wedge d\bar{z}_\beta$$

REMARK: ω_1, ω kähler forms.

this is a global expression:

$$(1) \quad Ric(\omega_1) = Ric(\omega) - \underbrace{dd^c}_{=\frac{i}{\pi}\partial\bar{\partial}} \log\left(\frac{\omega_1^n}{\omega^n}\right)$$

From (1) we can see that the cohomology class of $Ric(\omega_1)$ is $\{Ric(\omega_1)\} = \{Ric(\omega)\}$

Moreover, $\{Ric(\omega_1)\} = c_1(X) = c_1(K_X^{-1}) = -c_1(K_X)$

chern class

One can ask: Give $\eta \in c_1(X)$, a real (1,1) form, does $\exists\ \omega$ a Kahler form on X such that $Ric(\omega) = \eta$? (Known as Calabi Conjecture '54)

CALABI-YAU METHOD

Fix $\alpha \in H^{1,1}(X, \mathbb{R})$ Kahler class and pick a representative $\omega \in \alpha$ kahler form

From $\delta\delta$-lemma $\Rightarrow$ $Ric(\omega) = \eta + dd^c h$

where $h \in C^\infty(X)$.

We search for

$$\omega_\varphi := \omega + dd^c \varphi \in \alpha$$

$$Ric(\omega_\varphi) = \eta$$

Using (1) the Calabi problem is equivalent to studying the following Monge-Ampère (MA) eqn:

$$MA(\varphi) := (\omega + dd^c \varphi)^n = \underbrace{e^h}_{=: f \text{ (density)}} \cdot \omega^n \qquad (*)$$

Now we will write $(*)$ in local coordinates

$$\frac{\det\left(\omega_{\alpha\beta} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}\right)}{\det \omega_{\alpha\beta}} = f$$

REMARK: Eqn $(*)$ carries a compatibility criterion

$$\int_X f\omega^n = \int_X \omega^n$$

the Calabi conjecture was solved by Yau in 78

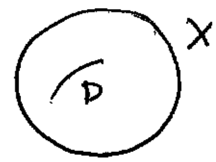
THM If f is strictly positive and $f \in C^\infty(X)$

$\Rightarrow \exists! \varphi \in C^\infty(X)$ solution of $(*)$ where uniqueness is up to an additive constant.

Consider $D \subset X$ is a closed subset, look at

$$MA(\varphi) = f\omega^n$$

But $f \in C^\infty(X \setminus D)$.



Ask: $*$ Guedj – Zeriahi '07 $\left.\right\}$ $\exists!$ a "weak" solution of $\circledast$?
$*$ Drew '09

<u>Q</u>: What can we say about the regularity of the solution φ? Moreover what can we say about its asymptotic behavior?

THEOREM A (– Lu '14) "C^0-estimate"

If $f \lesssim e^{-\Phi}$ where Φ is a quasi-plurisubharmonic function (locally = smooth + psh)

$\Rightarrow$ $\forall a > 0, \exists A > 0$ UNIFORM (doesn't depend on φ)

$$\varphi \geq a\Phi - A$$

REMARK Here we don't ask for regularity for f.

History: In '98 Kolodziej: generalization of C^0 estimates in Yau's proof

when $e^{-\Phi} \in L^p$ $p > 1 \Rightarrow \varphi$ is bdd on X

But $e^{-\Phi} \notin L^1$ (a priori)

In general the solution is not embedded.

THEOREM B (– Lu '14) Elliptic regularity thm

If $0 < f \in C^\infty(X \setminus D)$ and $f = e^{\psi^+ - \psi^-}$

where $\psi^\pm$ are qpsh and $\psi^- \in L^\infty_{loc}(X \setminus D)$

$\Rightarrow$ $\exists!$ solution φ of $*$ smooth on $X \setminus D$ uniqueness up to a constant.

<u>Motivation</u>: Minimal Model Program in birational geometry. In particular to the study of projective varieties with log canonical singularities (very general class of singularities in MMP).

* Berman - Guenoscia '14: existence of KE metrics on general type varieties with log canonical singularities. Their problem reduced to studying

$$MA(\varphi) = e^{\varphi} g \, \underline{dV} \quad (**)$$

$g \sim \frac{1}{|S_D|^2}$ where S_D is the defining section of a divisor $D \subset X$

If $\exists$ a solution φ of $(**)$ then we can apply thm B with $\psi^+ = \varphi$, $\psi^- = 2 \log |S_D|$

B-G use a variational approach
N-L plutipotential method

Idea of proof of thm B

$$MA(\varphi) = f\omega^n = e^{\psi^+ - \psi^-} \omega^n$$

$\boxed{\text{Step 0}}$: Demally's reg thm to regularize $\psi^{\pm} \rightsquigarrow \psi^{\pm}_{\epsilon}$ (smooth and qpsh)

Thanks to Yaus thm, $\exists ! \; \varphi_\epsilon \in C^\infty(X)$ solution $(\sup_X \varphi_\epsilon = 0)$ of

$$MA(\varphi_\epsilon) = C_\epsilon \, e^{\psi_\epsilon^+ - \psi_\epsilon^-} \omega^n$$

where $\int_X \omega^n = \int_X c_\epsilon e^{\psi_\epsilon^+ - \psi_\epsilon^-} \omega^n$

Need uniform estimates for φ_ϵ

$\boxed{\text{Step 1}}$ Obtain C^0-estimate

$f = e^{\psi^+ - \psi^-}$ by assumption, but actually

$f = e^{\psi^+ - \psi^-} \leq Ce^{-\psi^-}$ (apply thm A w/ $\Phi = \psi^-$)

and I get $\varphi \gg a\psi^- - A$.

$\boxed{\text{Step 2}}$ Obtain Laplacian estimate

$$\Delta_\omega \varphi \leq A e^{-\psi^-}$$

Note step 1 + step 2 $\Rightarrow$ uniform "C^0 and C^2" estimate for φ_ϵ on each $K \subset X \setminus D$ since $\psi^- \in L^\infty_{Loc}(X \setminus D)$

then the game is over.

$\boxed{\text{Step 3}}$ We can apply Evans-Krylov theory, Schauder estimates + bootstrap estimate

$$\|\varphi_\epsilon\|_{C^{k,\beta}(K)} \leq C_{k,\beta,K} \qquad k \geq 2,\ \beta \in (0,1)$$

$\boxed{\text{Step 4}}$ Apply Ascoli-Arzela

$\varphi_\epsilon \longrightarrow \varphi$ in $C^{k,\beta}(K)$ $\forall$ $k \geq 2$ $\beta \in (0,1)$

$\Rightarrow$ φ is smooth on $X \setminus D$

$\square$

EXAMPLES:

Consider $f = \dfrac{h}{|S_D|^2 (-\log |S_D|)^{1+\alpha}}$

where D-divisor, S_D- Defining section, $h \in C^\infty(X)$, $0 < \alpha \in \mathbb{R}$. We can apply THM B

$\psi^+ = \log h - (1+\alpha) \log(-\log |S_D|)$

$\psi^- = 2 \log |S_D|$

Now the question: What about the asymptotic behavior of the solution φ?

Kolodziej '98 recovers the case of such densities for $\alpha > n$ φ is globally bounded on X. With new methods, (-Lu) $\alpha > 1$, If $\alpha \leq 1 \Rightarrow \varphi$ is not bounded on X and give precise upper and lower bounds.

$\boxed{\alpha = 1}$ Poincaré case:

$$-\log(-\log|S_D|) - c \leq \varphi \leq -\log(-\log|S_D|)^q + c$$

$$\forall q \in (0,1)$$

Auray '14 with strong condition h, $\varphi \simeq -\log(-\log|S_D|)$

Pluripotencial methods (used to prove thm A)

Monge-Ampère capacity of a Borel set $E \subset X$

$$\mathrm{Cap}_\omega(E) := \sup\left\{ \int_E (\omega + dd^c u)^n \ \ \text{qpsh and } -1 < u \leq 0 \right\}$$

$\mathrm{Cap}_\omega(E) = 0 \Leftrightarrow E$ is pluripolar

$\Downarrow$

$\mathrm{Vol}_\omega(E) = 0 \qquad E = \{g = -\infty\} \quad g$ is q psh

Kolodziej's approach. Goal: $\exists\, T_\infty > 0$

$$\mathrm{Cap}_\omega(\{\varphi < -t\}) = 0 \quad \forall\, t \geq T_\infty$$

$\Rightarrow \exists\, C > 0 : \ \varphi \geq -C$

$\Rightarrow$ the solution φ is bdd

- Lu : New tool: Generalized Capacities

Take ψ qpsh function

$$\mathrm{Cap}_\psi(E) = \sup\left\{ \int_E (\omega + dd^c u)^n,\ u \text{ qpsh},\ \psi - 1 \leq u \leq \psi \right\}$$

Goal: $\exists\, T_\infty > 0$

$$\mathrm{Cap}_\psi(\{\varphi < \psi - t\}) = 0 \quad \forall\, t \geq \_\_$$

$\Rightarrow \exists\, C > 0 \quad \varphi \geq \psi - C.$